Question 1. *Let*

$$u_1 = (1,2,1), \qquad u_2 = (-2,-4,-2), \qquad u_3 = (0,2,3),$$
$$u_4 = (2,0,-3), \qquad u_5 = (-3,8,16),$$

and let $\mathbf{b} = (2,6,8)$. *check* $\mathbf{b}$ *can be expressed as a linear combination of* u_1, u_2, u_3, u_4, u_5.

Xingjian's Solution: **Xingjian comment:**this question is not a proof. This is just a small check. By definition, $\mathbf{b}$ can be expressed as a linear combination of u_1, u_2, u_3, u_4, u_5 if there exist scalars a_1, a_2, a_3, a_4, a_5 such that

$$(2,6,8) = a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5.$$

Substituting the given vectors, we obtain

$$(2,6,8) = a_1(1,2,1) + a_2(-2,-4,-2) + a_3(0,2,3) + a_4(2,0,-3) + a_5(-3,8,16).$$

Equating components, this is equivalent to the linear system

$$\begin{cases} a_1 - 2a_2 + 2a_4 - 3a_5 = 2, \\ 2a_1 - 4a_2 + 2a_3 + 8a_5 = 6, \\ a_1 - 2a_2 + 3a_3 - 3a_4 + 16a_5 = 8. \end{cases} \tag{1}$$

Then for the solving we will use this following way:
Eliminate a_1 from the second and third equations. **Xingjian comment:**Actually you don't need to write all the calculation in paper. you can directly write system (2). I am just here for showing the details.

We perform the row operations

$$\mathrm{Eq}_2 \leftarrow \mathrm{Eq}_2 - 2\,\mathrm{Eq}_1, \qquad \mathrm{Eq}_3 \leftarrow \mathrm{Eq}_3 - \mathrm{Eq}_1.$$

For the second equation:

$$\begin{aligned} \mathrm{Eq}_2 - 2\mathrm{Eq}_1 &= \big(2a_1 - 4a_2 + 2a_3 + 0a_4 + 8a_5\big) - 2\big(a_1 - 2a_2 + 0a_3 + 2a_4 - 3a_5\big) \\ &= (2a_1 - 2a_1) + (-4a_2 + 4a_2) + (2a_3 - 0a_3) + (0a_4 - 4a_4) + (8a_5 + 6a_5) \\ &= 2a_3 - 4a_4 + 14a_5. \end{aligned}$$

The right–hand side becomes $6 - 2 \cdot 2 = 2$.

For the third equation:

$$\begin{aligned} \mathrm{Eq}_3 - \mathrm{Eq}_1 &= \big(a_1 - 2a_2 + 3a_3 - 3a_4 + 16a_5\big) - \big(a_1 - 2a_2 + 0a_3 + 2a_4 - 3a_5\big) \\ &= (a_1 - a_1) + (-2a_2 + 2a_2) + (3a_3 - 0a_3) + (-3a_4 - 2a_4) + (16a_5 + 3a_5) \\ &= 3a_3 - 5a_4 + 19a_5. \end{aligned}$$

The right–hand side becomes $8 - 2 = 6$.

Thus the system (1) is equivalent to

$$\begin{aligned} a_1 - 2a_2 \qquad + 2a_4 - 3a_5 &= 2, \\ 2a_3 - 4a_4 + 14a_5 &= 2, \\ 3a_3 - 5a_4 + 19a_5 &= 6. \end{aligned} \tag{2}$$

Multiply the second equation by $\frac{1}{2}$:

$$\mathrm{Eq}_2 \leftarrow \tfrac{1}{2}\,\mathrm{Eq}_2,$$

we can get

$$a_3 - 2a_4 + 7a_5 = 1.$$

So the system becomes

$$\begin{aligned} a_1 - 2a_2 \qquad + 2a_4 - 3a_5 &= 2, \\ a_3 - 2a_4 + 7a_5 &= 1, \\ 3a_3 - 5a_4 + 19a_5 &= 6. \end{aligned} \tag{3}$$

Eliminate a_3 from the third equation.

Now perform

$$\mathrm{Eq}_3 \leftarrow \mathrm{Eq}_3 - 3\,\mathrm{Eq}_2.$$

Compute:

$$\begin{aligned} \mathrm{Eq}_3 - 3\mathrm{Eq}_2 &= \big(3a_3 - 5a_4 + 19a_5\big) - 3\big(a_3 - 2a_4 + 7a_5\big) \\ &= (3a_3 - 3a_3) + (-5a_4 + 6a_4) + (19a_5 - 21a_5) \\ &= a_4 - 2a_5. \end{aligned}$$

The right–hand side becomes $6 - 3 \cdot 1 = 3$.

Thus the final equivalent system is

$$\begin{aligned} a_1 - 2a_2 \qquad + 2a_4 - 3a_5 &= 2, \\ a_3 - 2a_4 + 7a_5 &= 1, \\ a_4 - 2a_5 &= 3. \end{aligned} \tag{4}$$

Then for the system (4), **we eliminate a_4 from the first and second equations.** We perform the row operations:

$$\mathrm{Eq}_1 \leftarrow \mathrm{Eq}_1 - 2\,\mathrm{Eq}_3, \qquad \mathrm{Eq}_2 \leftarrow \mathrm{Eq}_2 + 2\mathrm{Eq}_3.$$

For the first equation:

$$\begin{aligned} \mathrm{Eq}_1 - 2\mathrm{Eq}_3 &= \big(a_1 - 2a_2 + 2a_4 - 3a_5\big) - 2\big(a_4 - 2a_5\big) \\ &= (a_1 - 0a_1) + (-2a_2 - 0a_2) + (2a_4 - 2a_4) + (-3a_5 + 4a_5) \\ &= a_1 - 2a_2 + a_5. \end{aligned}$$

The right–hand side becomes $2 - 3 \cdot 2 = -4$.

For the second equation:

$$\begin{aligned}\mathrm{Eq}_2 + 2\mathrm{Eq}_3 &= \big(a_3 - 2a_4 + 7a_5\big) + 2\big(a_4 - 2a_5\big)\\ &= a_3 + (-2a_4 + 2a_4) + (7a_5 - 4a_5)\\ &= a_3 + 3a_5.\end{aligned}$$

The right–hand side becomes $1 + 3 \cdot 2 = 7$.

So the system becomes

$$\begin{aligned} a_1 - 2a_2 \qquad\quad &+ a_5 = -4,\\ a_3 \qquad\quad &+ 3a_5 = 7,\\ a_4 - 2a_5 &= 3.\end{aligned} \tag{5}$$

We rearrange (5), then we can get the relation:

$$\begin{aligned} a_1 &= 2a_2 - a_5 - 4,\\ a_3 &= -3a_5 + 7,\\ a_4 &= 2a_5 + 3.\end{aligned} \tag{6}$$

for any choice of scalar a_2 and a_5, we can take the vector of the form:

$$(a_1, a_2, a_3, a_4, a_5) = (2a_2 - a_5 - 4, a_2, -3a_5 + 7, 2a_5 + 3, a_5).$$

In particular, let $a_2 = 0, a_5 = 0$, we get a particular solution $(a_1, a_2, a_3, a_4, a_5) = (-4, 0, 7, 3, 0)$ and we can compute here

$$-4 \times (1, 2, 1) + 0 \times (-2, -4, -2) + 7 \times (0, 2, 3) + 3 \times (2, 0, -3) + 0 \times (-3, 8, 16)$$

$$= (-4, -8, -4) + (0, 14, 21) + (6, 0, 6) = (2, 6, 8).$$

Thus it is indeed a linear combination. we finished the small check!

Xingjian comment: Exam 1 will never test this complicated algebraic calculation like this check! You just need to understand the idea.

Xingjian Xu
xingjianxu@ufl.edu
LIT 453
Webpage: https://people.clas.ufl.edu/xingjianxu/mas4105-26springs/
Office Hours: Tuesday, Periods 3 and 5, LIT 453