Theorem 1. *Let*

$$S = \{\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2 : \ s, t \in \mathbb{R}\} \subset \mathbb{R}^3.$$

Then S can be written as a single linear equation

$$ax + by + cz = d \quad \textit{with } (a, b, c) \neq (0, 0, 0)$$

if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, i.e.

$$\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}.$$

Proof. *($\Rightarrow$)* Assume there exists a nonzero vector $\mathbf{n} = (a, b, c)$ and a scalar d such that

$$S = \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{n} \cdot \mathbf{x} = d\}.$$

Since $\mathbf{x}_0 \in S$, we have $\mathbf{n} \cdot \mathbf{x}_0 = d$. For any $\mathbf{x} \in S$,

$$\mathbf{x} = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2,$$

and thus

$$\mathbf{n} \cdot (\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2) = d.$$

Subtracting $\mathbf{n} \cdot \mathbf{x}_0 = d$, we obtain

$$s(\mathbf{n} \cdot \mathbf{v}_1) + t(\mathbf{n} \cdot \mathbf{v}_2) = 0 \quad \text{for all } s, t \in \mathbb{R}.$$

This implies

$$\mathbf{n} \cdot \mathbf{v}_1 = 0, \qquad \mathbf{n} \cdot \mathbf{v}_2 = 0.$$

Hence both $\mathbf{v}_1$ and $\mathbf{v}_2$ lie in the plane orthogonal to $\mathbf{n}$. If $\mathbf{v}_1$ and $\mathbf{v}_2$ were linearly dependent, then S would be a line, contradicting the fact that $\{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = d\}$ is a plane. Therefore $\mathbf{v}_1$ and $\mathbf{v}_2$ must be linearly independent, and so

$$\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}.$$

($\Leftarrow$) Assume $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$. Define

$$\mathbf{n} := \mathbf{v}_1 \times \mathbf{v}_2.$$

Then $\mathbf{n} \neq \mathbf{0}$ and

$$\mathbf{n} \cdot \mathbf{v}_1 = \mathbf{n} \cdot \mathbf{v}_2 = 0.$$

Let $d := \mathbf{n} \cdot \mathbf{x}_0$. For any $\mathbf{x} \in S$,

$$\mathbf{x} = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2,$$

and thus

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{x}_0 + s(\mathbf{n} \cdot \mathbf{v}_1) + t(\mathbf{n} \cdot \mathbf{v}_2) = d.$$

Hence $S \subseteq \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = d\}$.

Conversely, if $\mathbf{x} \in \mathbb{R}^3$ satisfies $\mathbf{n} \cdot \mathbf{x} = d$, then

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0,$$

so $\mathbf{x} - \mathbf{x}_0$ lies in the two-dimensional subspace orthogonal to $\mathbf{n}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and both orthogonal to $\mathbf{n}$, they form a basis of this subspace. Therefore there exist $s, t \in \mathbb{R}$ such that

$$\mathbf{x} - \mathbf{x}_0 = s\mathbf{v}_1 + t\mathbf{v}_2,$$

and hence $\mathbf{x} \in S$. This shows

$$\{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = d\} \subseteq S.$$

Combining both inclusions yields

$$S = \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = d\},$$

which completes the proof. □

Example 1 (Counterexample)**.** *Let*

$$\mathbf{x} = (1, 2, 3) + s(1, 2, 3) + t(2, 4, 6).$$

Since $(2, 4, 6) = 2(1, 2, 3)$*, the direction vectors are linearly dependent, and* S *is a line. A line in* $\mathbb{R}^3$ *cannot be represented by a single nontrivial linear equation* $ax + by + cz = d$*, which always defines a plane.*

Theorem 2. *Let*

$$S = \left\{ \mathbf{x}_0 + \sum_{i=1}^{k} t_i \mathbf{v}_i : \ t_i \in \mathbb{R} \right\} \subset \mathbb{R}^n.$$

Then S *can be written as the solution set of exactly* m *independent linear equations if and only if*

$$m = n - \operatorname{rank}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}.$$

In particular, S *is the solution of a single linear equation if and only if* $\operatorname{rank}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = n - 1$*.*

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