Question 1: Label the following statements as true or false.

(a). Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

(b). If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.

(c). There exists a square matrix with no eigenvectors.

(d). Eigenvalues must be nonzero scalars.

(e). Any two eigenvectors are linearly independent.

(f). The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

(g). Linear operators on infinite-dimensional vector spaces never have eigenvalues.

(h). An $n \times n$ matrix A with entries from a field F is similar to a diagonal matrix if and only if there is a basis for F^n consisting of eigenvectors of A.

(i). Similar matrices always have the same eigenvalues.

(j). Similar matrices always have the same eigenvectors.

(k). The sum of two eigenvectors of an operator T is always an eigenvector of T.

Xingjian's Solution:

(a). **False.** A linear operator need not have any eigenvalues over the given field, and even if it does, they need not be distinct.

(b). **True.** If v is an eigenvector, then every nonzero scalar multiple cv is also an eigenvector, so there are infinitely many.

(c). **True.** For example, over $\mathbb{R}$, the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no real eigenvectors.

(d). **False.** Zero can be an eigenvalue.

(e). **False.** Two eigenvectors corresponding to the same eigenvalue may be scalar multiples of each other.

(f). **False.** In general, the sum of two eigenvalues need not be an eigenvalue.

(g). **False.** Infinite-dimensional operators may still have eigenvalues.

(h). **True.** This is exactly the characterization of diagonalizability.

(i). **True.** Similar matrices have the same characteristic polynomial, hence the same eigenvalues.

(j). **False.** Similar matrices do not necessarily have the same eigenvectors.

(k). **False.** The sum of two eigenvectors need not be an eigenvector unless they correspond to the same eigenvalue and the sum is nonzero.

Question 3: For each of the following matrices $A \in M_{n\times n}(F)$:

(i). Determine all the eigenvalues of A.

(ii). For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii). If possible, find a basis for F^n consisting of eigenvectors of A.

(iv). If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \qquad \text{for } F = \mathbb{R}$$

(b)

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \qquad \text{for } F = \mathbb{R}$$

(c)

$$A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \qquad \text{for } F = \mathbb{C}$$

(d)

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \qquad \text{for } F = \mathbb{R}$$

Xingjian's Solution:

(a) Its characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4.$$

Hence $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$, so the eigenvalues are $\lambda = 4, \lambda = -1$.

For $\lambda = 4$, solve $(A - 4I)x = 0$:

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

This gives $-3x + 2y = 0$, so $y = \frac{3}{2}x$. Thus the eigenspace is

$$E_4 = \text{span}\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}.$$

For $\lambda = -1$, solve $(A + I)x = 0$:

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

This gives $x + y = 0$, so $y = -x$. Thus the eigenspace is

$$E_{-1} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Since there are two linearly independent eigenvectors, a basis of $\mathbb{R}^2$ consisting of eigenvectors is

$$\beta = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Let

$$Q = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $Q^{-1}AQ = D$.

(b) The characteristic polynomial is $\det(A - \lambda I) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$. Hence the eigenvalues are $\lambda = 1, 2, 3$.

For $\lambda = 1$, solve $(A - I)x = 0$:

$$A - I = \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix}.$$

A nonzero solution is

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

so

$$E_1 = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda = 2$, solve $(A - 2I)x = 0$:

$$A - 2I = \begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix}.$$

A nonzero solution is

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

so

$$E_2 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

For $\lambda = 3$, solve $(A - 3I)x = 0$:

$$A - 3I = \begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix}.$$

A nonzero solution is

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

so

$$E_3 = \text{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Since these three eigenvalues are distinct, the corresponding eigenvectors are linearly independent. Thus a basis of $\mathbb{R}^3$ consisting of eigenvectors is

$$\beta = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Let

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then $Q^{-1}AQ = D$.

(c) The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} i - \lambda & 1 \\ 2 & -i - \lambda \end{vmatrix} = (i - \lambda)(-i - \lambda) - 2.$$

Since $(i - \lambda)(-i - \lambda) = \lambda^2 + 1$, we get $\det(A - \lambda I) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$. Hence the eigenvalues are $\lambda = 1, -1$.

For $\lambda = 1$, solve $(A - I)x = 0$:

$$\begin{pmatrix} i - 1 & 1 \\ 2 & -i - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

From $(i - 1)x + y = 0$, we get $y = (1 - i)x$.

Taking $x = 1 + i$, we obtain $y = 2$. Hence one eigenvector is

$$\begin{pmatrix} 1 + i \\ 2 \end{pmatrix},$$

so

$$E_1 = \text{span} \left\{ \begin{pmatrix} 1 + i \\ 2 \end{pmatrix} \right\}.$$

For $\lambda = -1$, solve $(A + I)x = 0$:

$$\begin{pmatrix} i + 1 & 1 \\ 2 & 1 - i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

From $(i + 1)x + y = 0$, we get $y = -(1 + i)x$.

Taking $x = 1 - i$, we obtain $y = -2$. Hence one eigenvector is

$$\begin{pmatrix} 1 - i \\ -2 \end{pmatrix},$$

so

$$E_{-1} = \text{span}\left\{ \begin{pmatrix} 1-i \\ -2 \end{pmatrix} \right\}.$$

Therefore a basis of $\mathbb{C}^2$ consisting of eigenvectors is

$$\beta = \left\{ \begin{pmatrix} 1+i \\ 2 \end{pmatrix}, \begin{pmatrix} 1-i \\ -2 \end{pmatrix} \right\}.$$

Let

$$Q = \begin{pmatrix} 1+i & 1-i \\ 2 & -2 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $Q^{-1}AQ = D$.

(d) The characteristic polynomial is $\det(A - \lambda I) = \lambda(\lambda - 1)^2$. Hence the eigenvalues are $\lambda = 0, 1$.

For $\lambda = 0$, solve $Ax = 0$:

$$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

From $2x - z = 0$, we get $z = 2x$. Then $4x + y - 4z = 0$ gives

$$4x + y - 8x = 0 \quad \Rightarrow \quad y = 4x.$$

So

$$E_0 = \text{span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}.$$

For $\lambda = 1$, solve $(A - I)x = 0$:

$$A - I = \begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix}.$$

This gives $x - z = 0$, so $x = z$, with y free. Thus

$$E_1 = \left\{ \begin{pmatrix} x \\ y \\ x \end{pmatrix} : x, y \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Therefore we obtain three linearly independent eigenvectors, so a basis of $\mathbb{R}^3$ consisting of eigenvectors is

$$\beta = \left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Let

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $Q^{-1}AQ = D$.

Question 4: For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

(a). $V = \mathbb{R}^2$ and $T(a, b) = (-2a + 3b,\ -10a + 9b)$.

Xingjian's Solution:

Relative to the standard basis, the matrix is

$$A = \begin{pmatrix} -2 & 3 \\ -10 & 9 \end{pmatrix}.$$

Its characteristic polynomial is $\lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4)$. So the eigenvalues are $3, 4$.

For $\lambda = 3$, an eigenvector is $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$. For $\lambda = 4$, an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Thus one diagonalizing basis is

$$\beta = \left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}, \qquad [T]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}.$$

(b). $V = \mathbb{R}^3$ and $T(a, b, c) = (7a - 4b + 10c,\ 3a - 3b + 4c,\ -2a + b - 2c)$.

Xingjian's Solution:

Relative to the standard basis,

$$A = \begin{pmatrix} 7 & -4 & 10 \\ 3 & -3 & 4 \\ -2 & 1 & -2 \end{pmatrix}.$$

Its characteristic polynomial is $\lambda^3 - 2\lambda^2 - \lambda + 8$. This polynomial has only one real root, so T does not have a basis of real eigenvectors. Hence T is not diagonalizable over $\mathbb{R}$.

(c). $V = \mathbb{R}^3$ and $T(a, b, c) = (-4a + 3b - 6c,\ 7b + 12c,\ 6a - 6b + 11c)$.

Xingjian's Solution:

Relative to the standard basis,

$$A = \begin{pmatrix} -4 & 3 & -6 \\ 0 & 7 & 12 \\ 6 & -6 & 11 \end{pmatrix}.$$

Its characteristic polynomial is $(\lambda + 1)(\lambda^2 - 15\lambda + 128)$. The quadratic factor has no real roots, so there is only one real eigenvalue. Hence T is not diagonalizable over $\mathbb{R}$.

(d). $V = P_1(\mathbb{R})$ and $T(ax + b) = (-6a + 2b)x + (-6a + b)$.

Xingjian's Solution:

With respect to the basis $\{x, 1\}$,

$$[T]_{\{x,1\}} = \begin{pmatrix} -6 & 2 \\ -6 & 1 \end{pmatrix}.$$

Its characteristic polynomial is $(\lambda+2)(\lambda+3)$. So the eigenvalues are $-2, -3$.

For $\lambda = -2$, one eigenvector is $x + 2$.

For $\lambda = -3$, one eigenvector is $2x + 3$.

Hence

$$\beta = \{x+2,\ 2x+3\}, \qquad [T]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

(e). $V = P_2(\mathbb{R})$ and $T(f(x)) = xf'(x) + f(2)x + f(3)$.

Xingjian's Solution:

Let $f(x) = ax^2+bx+c$. Then $f'(x) = 2ax+b$, $f(2) = 4a+2b+c$, $f(3) = 9a+3b+c$. So

$$T(f) = 2ax^2 + (4a+3b+c)x + (9a+3b+c).$$

Relative to $\{x^2, x, 1\}$,

$$[T] = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 3 & 1 \\ 9 & 3 & 1 \end{pmatrix}.$$

Its eigenvalues are $0, 2, 4$.

Corresponding eigenvectors may be taken as

$$\lambda = 0:\ -x+3, \qquad \lambda = 2:\ -4x^2+13x+3, \qquad \lambda = 4:\ x+1.$$

Thus

$$\beta = \{-x+3,\ -4x^2+13x+3,\ x+1\}, \qquad [T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

(f). $V = P_3(\mathbb{R})$ and $T(f(x)) = f(x) + f(2)x$.

Xingjian's Solution:

Let $f(x) = ax^3+bx^2+cx+d$. Then $T(f) = f(x)+f(2)x = ax^3+bx^2+(8a+4b+3c+d)x+d$. The eigenvalues are 1 and 3.

A basis for the eigenspace corresponding to $\lambda = 1$ is

$$\{-x^3+2x^2,\ -x^3+4x,\ -x^3+8\},$$

and an eigenvector for $\lambda = 3$ is x. Therefore one diagonalizing basis is

$$\beta = \{-x^3+2x^2,\ -x^3+4x,\ -x^3+8,\ x\},$$

with

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(g). $V = P_3(\mathbb{R})$ and $T(f(x)) = xf'(x) + f''(x) - f(2)$.
Xingjian's Solution:
Let $f(x) = ax^3 + bx^2 + cx + d$. Then

$$f'(x) = 3ax^2 + 2bx + c, \qquad f''(x) = 6ax + 2b, \qquad f(2) = 8a + 4b + 2c + d.$$

Hence

$$T(f) = 3ax^3 + 2bx^2 + (6a + c)x + (-8a - 2b - 2c - d).$$

The eigenvalues are $-1,\ 1,\ 2,\ 3$. Corresponding eigenvectors may be taken as $\lambda = -1:\ 1$, $\lambda = 1:\ 1 - x$, $\lambda = 2:\ -3x^2 + 2$, $\lambda = 3:\ -2x^3 - 6x + 7$. Thus

$$\beta = \{1,\ 1 - x,\ -3x^2 + 2,\ -2x^3 - 6x + 7\},$$

and

$$[T]_\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(h). $V = M_{2\times 2}(\mathbb{R})$ and $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$.
Xingjian's Solution:
We have

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

This swaps the diagonal entries and leaves the off-diagonal entries fixed.

Eigenvalue 1 has eigenspace

$$\left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \right\},$$

so a basis is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Eigenvalue -1 has eigenvector

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence one diagonalizing basis is

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

with

$$[T]_\beta = \mathrm{diag}(1, 1, 1, -1).$$

(i). $V = M_{2\times 2}(\mathbb{R})$ and $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$.

Xingjian's Solution:
We have

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix},$$

so T swaps the two rows.

Eigenvalue 1 corresponds to matrices with equal rows:

$$\begin{pmatrix} a & b \\ a & b \end{pmatrix}.$$

A basis is

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Eigenvalue -1 corresponds to matrices with opposite rows:

$$\begin{pmatrix} a & b \\ -a & -b \end{pmatrix}.$$

A basis is

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Thus one diagonalizing basis is

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\},$$

and

$$[T]_\beta = \operatorname{diag}(1, 1, -1, -1).$$

(j). $V = M_{2\times 2}(\mathbb{R})$ and $T(A) = A^t + 2 \cdot \operatorname{tr}(A) \cdot I_2$.

Xingjian's Solution:
Let $T(A) = A^t + 2\ \operatorname{tr}(A)\, I_2$. Decompose $M_{2\times 2}(\mathbb{R})$ as

$$\operatorname{span}\{I_2\} \oplus \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \oplus \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Then: $T(I_2) = 5I_2$, so 5 is an eigenvalue;

$$T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so 1 is an eigenvalue of multiplicity 2; and

$$T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so -1 is an eigenvalue.

Therefore one diagonalizing basis is

$$\beta = \left\{ I_2, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

and

$$[T]_\beta = \text{diag}(5, 1, 1, -1).$$

Question 5: Prove Theorem 5.4: Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.

Xingjian's Solution:

Proof. We prove both directions. Suppose first that v is an eigenvector of T corresponding to λ. By definition of eigenvector, we have $v \neq 0$ and $T(v) = \lambda v$.

Therefore, $(T - \lambda I)(v) = T(v) - \lambda I(v) = T(v) - \lambda v = 0$. Hence $v \in N(T - \lambda I)$. So if v is an eigenvector corresponding to λ, then $v \neq 0$ and $v \in N(T - \lambda I)$.

Conversely, suppose that $v \neq 0$ and $v \in N(T - \lambda I)$. Then $(T - \lambda I)(v) = 0$. Thus $T(v) - \lambda v = 0$, which implies $T(v) = \lambda v$. Since $v \neq 0$, this shows that v is an eigenvector of T corresponding to λ.

Therefore, a vector $v \in V$ is an eigenvector of T corresponding to λ if and only if

$$v \neq 0 \quad \text{and} \quad v \in N(T - \lambda I).$$

□

Question 6: Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$.

Xingjian's Solution

Proof. Recall that λ is an eigenvalue of T if and only if there exists a nonzero vector $v \in V$ such that $T(v) = \lambda v$. Let $[v]_\beta$ denote the coordinate vector of v relative to the ordered basis β.

Suppose first that λ is an eigenvalue of T. Then there exists $v \neq 0$ such that $T(v) = \lambda v$. Taking coordinates with respect to β, we obtain $[T(v)]_\beta = [\lambda v]_\beta = \lambda[v]_\beta$. By the definition of matrix representation, $[T]_\beta[v]_\beta = \lambda[v]_\beta$.

Since $v \neq 0$, its coordinate vector $[v]_\beta \neq 0$. Hence λ is an eigenvalue of $[T]_\beta$.

Conversely, suppose that λ is an eigenvalue of $[T]_\beta$. Then there exists a nonzero vector $x \in \mathbb{F}^n$ such that $[T]_\beta x = \lambda x$. Because β is a basis, there exists a unique vector $v \in V$ such that $[v]_\beta = x$. Since $x \neq 0$, we have $v \neq 0$. Then $[T(v)]_\beta = [T]_\beta[v]_\beta = [T]_\beta x = \lambda x = \lambda[v]_\beta = [\lambda v]_\beta$. Since coordinate representations are unique, it follows that $T(v) = \lambda v$. Thus λ is an eigenvalue of T.

Therefore, λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$. □

Question 7. Let T be a linear operator on a finite-dimensional vector space V. We define the determinant of T, denoted $\det(T)$, as follows. Choose any ordered basis β for V, and define $\det(T) = \det([T]_\beta)$.

(a) Prove that the preceding definition is independent of the choice of an ordered basis for V. That is, if β and γ are two ordered bases for V, then $\det([T]_\beta) = \det([T]_\gamma)$.

Xingjian's Solution

Proof. Let P be the change-of-coordinates matrix from γ-coordinates to β-coordinates. Then $[T]_\gamma = P^{-1}[T]_\beta P$. Taking determinants,

$$\det([T]_\gamma) = \det(P^{-1}[T]_\beta P) = \det(P^{-1})\det([T]_\beta)\det(P).$$

Since $\det(P^{-1})\det(P) = 1$, we get $\det([T]_\gamma) = \det([T]_\beta)$. Hence the definition of $\det(T)$ is independent of the choice of basis. □

(b) Prove that T is invertible if and only if $\det(T) \neq 0$.

Xingjian's Solution

Proof. Choose an ordered basis β for V. Then T is invertible if and only if $[T]_\beta$ is invertible. For matrices, a square matrix is invertible if and only if its determinant is nonzero. Hence

$$T \text{ is invertible} \iff [T]_\beta \text{ is invertible} \iff \det([T]_\beta) \neq 0.$$

By definition, $\det(T) = \det([T]_\beta)$, so T is invertible $\iff \det(T) \neq 0$. □

(c) Prove that if T is invertible, then $\det(T^{-1}) = [\det(T)]^{-1}$.

Xingjian's Solution

Proof. Choose an ordered basis β for V. Since T is invertible, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$. Therefore,

$$\det(T^{-1}) = \det([T^{-1}]_\beta) = \det\left(([T]_\beta)^{-1}\right) = \frac{1}{\det([T]_\beta)} = \frac{1}{\det(T)}.$$

Thus $\det(T^{-1}) = [\det(T)]^{-1}$. □

(d) Prove that if U is also a linear operator on V, then $\det(TU) = \det(T) \cdot \det(U)$.

Xingjian's Solution

Proof. Choose an ordered basis β for V. Then $[TU]_\beta = [T]_\beta[U]_\beta$. Hence

$$\det(TU) = \det([TU]_\beta) = \det([T]_\beta[U]_\beta) = \det([T]_\beta)\det([U]_\beta) = \det(T)\det(U).$$

□

(e) Prove that $\det(T - \lambda I_V) = \det([T]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

Xingjian's Solution

Proof. Since $[I_V]_\beta = I$, we have $[T - \lambda I_V]_\beta = [T]_\beta - \lambda[I_V]_\beta = [T]_\beta - \lambda I$. Therefore,

$$\det(T - \lambda I_V) = \det([T - \lambda I_V]_\beta) = \det([T]_\beta - \lambda I).$$

□

Question 8. (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

Xingjian's Solution

Proof. Recall that 0 is an eigenvalue of T if and only if there exists a nonzero vector $v \in V$ such that $T(v) = 0$. This happens exactly when $\ker(T)$ contains a nonzero vector, that is, $\ker(T) \neq \{0\}$.

Since V is finite-dimensional, T is invertible if and only if T is one-to-one, and T is one-to-one if and only if $\ker(T) = \{0\}$. Therefore,

$$T \text{ is invertible} \iff \ker(T) = \{0\} \iff 0 \text{ is not an eigenvalue of } T.$$

□

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Xingjian's Solution

Proof. Assume first that λ is an eigenvalue of T. Then there exists $v \neq 0$ such that $T(v) = \lambda v$. Since T is invertible, $\lambda \neq 0$ by part (a). Apply T^{-1} to both sides: $v = T^{-1}(\lambda v) = \lambda T^{-1}(v)$. Thus $T^{-1}(v) = \lambda^{-1} v$. So λ^{-1} is an eigenvalue of T^{-1}.

Conversely, suppose λ^{-1} is an eigenvalue of T^{-1}. Then there exists $v \neq 0$ such that $T^{-1}(v) = \lambda^{-1} v$. Apply T to both sides: $v = \lambda^{-1} T(v)$. Hence $T(v) = \lambda v$, so λ is an eigenvalue of T.

Therefore, λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}. □

(c) State and prove results analogous to (a) and (b) for matrices.

Xingjian's Solution

Proof. Let A be an $n \times n$ matrix.

Analog of (a): The matrix A is invertible if and only if 0 is not an eigenvalue of A.

Indeed, 0 is an eigenvalue of A if and only if there exists $x \neq 0$ such that $Ax = 0$. This means the homogeneous system $Ax = 0$ has a nontrivial solution, which occurs if and only if A is not invertible. Hence A is invertible $\iff$ 0 is not an eigenvalue of A.

Analog of (b): If A is invertible, then λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

If $Ax = \lambda x$ for some $x \neq 0$, then applying A^{-1} gives $x = \lambda A^{-1} x$, so $A^{-1} x = \lambda^{-1} x$. Thus λ^{-1} is an eigenvalue of A^{-1}.

Conversely, if $A^{-1} x = \lambda^{-1} x$ for some $x \neq 0$, then applying A gives $Ax = \lambda x$. So λ is an eigenvalue of A.

This proves the matrix versions of parts (a) and (b). □

Question 9. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

Xingjian's Solution

Proof. Let

$$M = \begin{pmatrix} m_{11} & * & \cdots & * \\ 0 & m_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{nn} \end{pmatrix}$$

be an upper triangular matrix. A scalar λ is an eigenvalue of M if and only if $\det(M-\lambda I) = 0$. But $M - \lambda I$ is also upper triangular, with diagonal entries $m_{11} - \lambda,\ m_{22} - \lambda,\ \ldots,\ m_{nn} - \lambda$. The determinant of a triangular matrix is the product of its diagonal entries, so

$$\det(M - \lambda I) = \prod_{i=1}^{n} (m_{ii} - \lambda).$$

Therefore,

$$\det(M - \lambda I) = 0 \iff m_{ii} - \lambda = 0 \text{ for some } i \iff \lambda = m_{ii} \text{ for some } i.$$

Hence the eigenvalues of M are exactly the diagonal entries of M. □

Question 10. Let V be a finite-dimensional vector space, and let λ be any scalar.
(a) For any ordered basis β for V, prove that $[\lambda I_V]_\beta = \lambda I$.

Xingjian's Solution

Proof. Let $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis for V. Since

$$(\lambda I_V)(v_j) = \lambda v_j \quad \text{for each } j = 1, \ldots, n,$$

the coordinate vector of $(\lambda I_V)(v_j)$ relative to β is

$$[(\lambda I_V)(v_j)]_\beta = \lambda e_j,$$

where e_j is the jth standard basis vector in $\mathbb{F}^n$.

Thus the jth column of $[\lambda I_V]_\beta$ is λe_j. Therefore,

$$[\lambda I_V]_\beta = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix} = \lambda I.$$

□

(b) Compute the characteristic polynomial of λI_V.

Xingjian's Solution

Proof. Let $\dim V = n$, and choose any ordered basis β for V. By part (a), $[\lambda I_V]_\beta = \lambda I$. Therefore the characteristic polynomial of λI_V is

$$p_{\lambda I_V}(t) = \det([\lambda I_V]_\beta - tI) = \det(\lambda I - tI) = \det((\lambda - t)I).$$

Since $(\lambda - t)I$ is a diagonal matrix with diagonal entries $\lambda - t$, its determinant is $(\lambda - t)^n$. Hence the characteristic polynomial is

$$p_{\lambda I_V}(t) = (\lambda - t)^n.$$

Equivalently, some books write it as $p_{\lambda I_V}(t) = (-1)^n(t - \lambda)^n$. □

(c) Show that λI_V is diagonalizable and has only one eigenvalue.
Xingjian's Solution

Proof. Since $[\lambda I_V]_\beta = \lambda I$ for every ordered basis β, the matrix representation of λI_V is already diagonal. Hence λI_V is diagonalizable.

Also, from part (b), its characteristic polynomial is $(\lambda - t)^n$, so the only root is $t = \lambda$. Therefore the only eigenvalue of λI_V is λ.

One can also see this directly: for every nonzero vector $v \in V$, $(\lambda I_V)(v) = \lambda v$. So every nonzero vector is an eigenvector with eigenvalue λ, and no other eigenvalue can occur. □

Question 11. A scalar matrix is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.
Xingjian's Solution

Proof. If A is similar to λI, then there exists an invertible matrix P such that $A = P(\lambda I)P^{-1}$. Since λI commutes with every matrix, $P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda I$. Hence $A = \lambda I$. □

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.
Xingjian's Solution

Proof. Let A be diagonalizable. Then A is similar to a diagonal matrix D: $A = PDP^{-1}$ for some invertible matrix P. Since A has only one eigenvalue, every diagonal entry of D must be that same eigenvalue, say λ. Thus $D = \lambda I$.

Therefore $A = P(\lambda I)P^{-1} = \lambda I$ by part (a). Hence A is a scalar matrix. □

(c) Prove that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable.
Xingjian's Solution

Proof. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since A is upper triangular, its only eigenvalue is 1, with algebraic multiplicity 2.

To find the eigenspace, solve $(A - I)x = 0$. Now

$$A - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So if $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then $(A - I)x = 0 \quad \Longrightarrow \quad x_2 = 0.$

Hence the eigenspace is

$$E_1 = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} : x_1 \in F \right\},$$

which has dimension 1.

A 2×2 matrix is diagonalizable only if it has two linearly independent eigenvectors. Since A has only one linearly independent eigenvector, it is not diagonalizable. □

Question 12. (a) Prove that similar matrices have the same characteristic polynomial.
Xingjian's Solution

Proof. Suppose A and B are similar. Then there exists an invertible matrix P such that $B = P^{-1}AP$. Then

$$B - tI = P^{-1}AP - tI = P^{-1}AP - tP^{-1}IP = P^{-1}(A - tI)P.$$

Taking determinants,

$$\det(B - tI) = \det\left(P^{-1}(A - tI)P\right) = \det(P^{-1})\det(A - tI)\det(P) = \det(A - tI).$$

Hence A and B have the same characteristic polynomial. □

(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.
Xingjian's Solution

Proof. Let T be a linear operator on V, and let β, γ be two ordered bases for V. Then the matrices $[T]_\beta$ and $[T]_\gamma$ are similar. By part (a), similar matrices have the same characteristic polynomial. Therefore

$$\det([T]_\beta - tI) = \det([T]_\gamma - tI).$$

Hence the characteristic polynomial of T does not depend on the choice of basis. □

Question 14. For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).
Xingjian's Solution

Proof. The characteristic polynomial of A is $p_A(t) = \det(A - tI)$. Now

$$p_{A^t}(t) = \det(A^t - tI) = \det\left((A - tI)^t\right).$$

Since the determinant of a matrix equals the determinant of its transpose,

$$\det\left((A - tI)^t\right) = \det(A - tI).$$

Thus $p_{A^t}(t) = p_A(t)$. Therefore A and A^t have the same characteristic polynomial, and hence the same eigenvalues. □

Question 15. **(a)** Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

Xingjian's Solution

Proof. Since x is an eigenvector of T corresponding to λ, we have $T(x) = \lambda x$. We prove by induction on m that $T^m(x) = \lambda^m x$.

For $m = 1$, this is exactly the definition of eigenvector.

Assume the statement holds for some $m \geq 1$. Then

$$T^{m+1}(x) = T(T^m(x)) = T(\lambda^m x) = \lambda^m T(x) = \lambda^m(\lambda x) = \lambda^{m+1}x.$$

Thus the result holds for all positive integers m.

Since $x \neq 0$, it follows that x is an eigenvector of T^m with eigenvalue λ^m. □

(b) State and prove the analogous result for matrices.

Xingjian's Solution

Proof. **Statement:** Let A be an $n \times n$ matrix, and let $x \neq 0$ be an eigenvector of A corresponding to the eigenvalue λ. Then for every positive integer m, x is an eigenvector of A^m corresponding to the eigenvalue λ^m.

The proof is the same as in part (a). Since $Ax = \lambda x$, we prove by induction that $A^m x = \lambda^m x$ for every positive integer m.

For $m = 1$, it is true. If $A^m x = \lambda^m x$, then

$$A^{m+1}x = A(A^m x) = A(\lambda^m x) = \lambda^m Ax = \lambda^m(\lambda x) = \lambda^{m+1}x.$$

Hence the statement is true for matrices. □

Question 17. Let T be the linear operator on $M_{n\times n}(\mathbb{R})$ defined by $T(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.

Xingjian's Solution

Proof. Suppose $A \neq 0$ is an eigenvector of T with eigenvalue λ. Then $T(A) = A^t = \lambda A$.

Apply transpose to both sides:

$$A = (A^t)^t = (\lambda A)^t = \lambda A^t.$$

Using $A^t = \lambda A$, we get

$$A = \lambda(\lambda A) = \lambda^2 A.$$

Hence

$$(\lambda^2 - 1)A = 0.$$

Since $A \neq 0$, it follows that $\lambda^2 - 1 = 0$, so $\lambda = \pm 1$.

Now we show both values occur. If A is symmetric and nonzero, then $A^t = A$, so A is an eigenvector with eigenvalue 1. If A is skew-symmetric and nonzero, then $A^t = -A$, so A is an eigenvector with eigenvalue -1.

Therefore the only eigenvalues of T are ± 1. □

(b) Describe the eigenspaces corresponding to each eigenvalue of T.
Xingjian's Solution

Proof. For $\lambda = 1$, the eigenspace is $E_1 = \{A \in M_{n\times n}(\mathbb{R}) : A^t = A\}$, the subspace of all symmetric matrices.

For $\lambda = -1$, the eigenspace is $E_{-1} = \{A \in M_{n\times n}(\mathbb{R}) : A^t = -A\}$, the subspace of all skew-symmetric matrices. Thus the eigenspaces are precisely the symmetric and skew-symmetric matrices. □

(c) Find an ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.
Xingjian's Solution

Proof. A basis of symmetric matrices in $M_{2\times 2}(\mathbb{R})$ is

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and a basis of skew-symmetric matrices is

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\beta = (S_1, S_2, S_3, K)$. Then

$$T(S_1) = S_1, \qquad T(S_2) = S_2, \qquad T(S_3) = S_3, \qquad T(K) = -K.$$

Hence each basis vector is an eigenvector, so $[T]_\beta = \operatorname{diag}(1, 1, 1, -1)$. □

(d) Find an ordered basis β for $M_{n\times n}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix for $n > 2$.
Xingjian's Solution

Proof. Let E_{ij} denote the matrix having 1 in the (i,j)-entry and 0 elsewhere.

A basis for the symmetric matrices is given by

$$E_{ii} \quad (1 \le i \le n), \qquad E_{ij} + E_{ji} \quad (1 \le i < j \le n).$$

A basis for the skew-symmetric matrices is given by $E_{ij} - E_{ji} \quad (1 \le i < j \le n)$. Take β to be the ordered basis consisting first of all the symmetric basis elements, then all the skew-symmetric basis elements. For each such basis element, $T(E_{ii}) = E_{ii}$,

$$T(E_{ij} + E_{ji}) = E_{ij} + E_{ji},$$

and

$$T(E_{ij} - E_{ji}) = -(E_{ij} - E_{ji}).$$

Thus every basis vector is an eigenvector, so $[T]_\beta$ is diagonal, with 1's corresponding to the symmetric basis vectors and -1's corresponding to the skew-symmetric basis vectors. □

Question 18. Let $A, B \in M_{n\times n}(\mathbb{C})$.

(a) Prove that if B is invertible, then there exists a scalar $c \in \mathbb{C}$ such that $A + cB$ is not invertible. *Hint: Examine* $\det(A + cB)$.

Xingjian's Solution

Proof. Since B is invertible, $A + cB = B(B^{-1}A + cI)$. Therefore

$$\det(A + cB) = \det(B)\det(B^{-1}A + cI).$$

Because $\det(B) \neq 0$, the matrix $A + cB$ is noninvertible exactly when $\det(B^{-1}A + cI) = 0$.

Now $\det(B^{-1}A + cI)$ is a polynomial in c of degree n. Over $\mathbb{C}$, every nonconstant polynomial has a root. Hence there exists $c \in \mathbb{C}$ such that $\det(B^{-1}A + cI) = 0$.

For this value of c, $\det(A + cB) = 0$, so $A + cB$ is not invertible. □

(b) Find nonzero 2×2 matrices A and B such that both A and $A + cB$ are invertible for all $c \in \mathbb{C}$.

Xingjian's Solution

Proof. Take

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Both A and B are nonzero. Also,

$$A + cB = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

Its determinant is $\det(A + cB) = 1$ for every $c \in \mathbb{C}$. Hence $A + cB$ is invertible for all c, and of course A is invertible. □

Question 19. Let A and B be similar $n \times n$ matrices. Prove that there exists an n-dimensional vector space V, a linear operator T on V, and ordered bases β and γ for V such that $A = [T]_\beta$ and $B = [T]_\gamma$. *Hint: Use Exercise 14 of Section 2.5.*

Xingjian's Solution

Proof. Since A and B are similar, there exists an invertible matrix Q such that $B = Q^{-1}AQ$. Let $V = F^n$, where F is the underlying field, and define a linear operator $T : V \to V$ by $T(x) = Ax$.

Let β be the standard ordered basis of F^n. Then clearly $[T]_\beta = A$.

Now let $\gamma = \{Q^{-1}e_1, \ldots, Q^{-1}e_n\}$, where $\{e_1, \ldots, e_n\}$ is the standard basis. Since Q is invertible, γ is an ordered basis of V.

By the change-of-basis formula, the matrix of T relative to γ is

$$[T]_\gamma = Q^{-1}[T]_\beta Q = Q^{-1}AQ = B.$$

Therefore there exist V, T, and ordered bases β, γ such that $A = [T]_\beta, B = [T]_\gamma$. □

Question 20. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

Xingjian's Solution

Proof. By definition, the characteristic polynomial is $f(t) = \det(A - tI)$.

Substituting $t = 0$, we obtain $f(0) = \det(A - 0I) = \det(A)$.

But from the given form of $f(t)$, $f(0) = a_0$. Hence $a_0 = \det(A)$.

Now A is invertible if and only if $\det(A) \neq 0$. Since $\det(A) = a_0$, we conclude that

$$A \text{ is invertible} \iff a_0 \neq 0.$$

□

Question 22. (a) Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients from F. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(T)(x) = g(\lambda)x$. That is, x is an eigenvector of $g(T)$ with corresponding eigenvalue $g(\lambda)$.

Xingjian's Solution

Proof. Let

$$g(t) = a_0 + a_1 t + \cdots + a_m t^m.$$

Then

$$g(T) = a_0 I + a_1 T + \cdots + a_m T^m.$$

Suppose x is an eigenvector of T with eigenvalue λ, so $T(x) = \lambda x$.

By Exercise 15(a), $T^k(x) = \lambda^k x \quad$ for all $k \geq 1$.

Therefore

$$g(T)(x) = (a_0 I + a_1 T + \cdots + a_m T^m)(x) = a_0 x + a_1 \lambda x + \cdots + a_m \lambda^m x.$$

Factor out x:

$$g(T)(x) = \big(a_0 + a_1 \lambda + \cdots + a_m \lambda^m\big) x = g(\lambda) x.$$

Hence x is an eigenvector of $g(T)$ with eigenvalue $g(\lambda)$. □

(b) State and prove a comparable result for matrices.
Xingjian's Solution

Proof. **Statement:** Let A be an $n \times n$ matrix over F, and let $g(t)$ be a polynomial over F. If $x \neq 0$ is an eigenvector of A with eigenvalue λ, then

$$g(A)x = g(\lambda)x.$$

Thus x is an eigenvector of $g(A)$ with eigenvalue $g(\lambda)$.

The proof is exactly the same as in part (a). Write

$$g(t) = a_0 + a_1 t + \cdots + a_m t^m.$$

Then

$$g(A) = a_0 I + a_1 A + \cdots + a_m A^m.$$

If $Ax = \lambda x$, then $A^k x = \lambda^k x$ for all $k \geq 1$. Hence

$$g(A)x = a_0 x + a_1 Ax + \cdots + a_m A^m x = a_0 x + a_1 \lambda x + \cdots + a_m \lambda^m x = g(\lambda)x.$$

□

(c) Verify (b) for the matrix A in Exercise 3(a) with polynomial $g(t) = 2t^2 - t + 1$, eigenvector

$$x = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

and corresponding eigenvalue $\lambda = 4$.
Xingjian's Solution

Proof. By part (b), if x is an eigenvector of A with eigenvalue $\lambda = 4$, then $g(A)x = g(4)x$.

Now compute $g(4) = 2(4)^2 - 4 + 1 = 32 - 4 + 1 = 29$. Therefore

$$g(A)x = 29x = 29 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 58 \\ 87 \end{pmatrix}.$$

So the result is verified. □

23. Use Exercise 22 to prove that if $f(t)$ is the characteristic polynomial of a diagonalizable linear operator T, then $f(T) = T_0$, the zero operator. (In Section 5.4 we prove that this result does not depend on the diagonalizability of T.)
Xingjian's Solution

Proof. Assume T is diagonalizable. Then V has a basis consisting of eigenvectors of T. Let $\beta = \{v_1, \ldots, v_n\}$ be such a basis, and suppose $T(v_i) = \lambda_i v_i \quad \text{for } i = 1, \ldots, n$.

Since $f(t)$ is the characteristic polynomial of T, each eigenvalue λ_i is a root of $f(t)$. Hence $f(\lambda_i) = 0 \quad \text{for all } i$.

By Exercise 22(a), $f(T)(v_i) = f(\lambda_i)v_i = 0 \cdot v_i = 0 \quad \text{for all } i$.

Thus $f(T)$ sends every basis vector v_i to 0. Therefore $f(T)$ is the zero operator: $f(T) = T_0$. □

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