Question 1: Label the following statements as true or false.

(a). If E is an elementary matrix, then $\det(E) = \pm 1$.

(b). For any $A, B \in M_{n\times n}(F)$, $\det(AB) = \det(A)\det(B)$.

(c). A matrix $M \in M_{n\times n}(F)$ is invertible if and only if $\det(M) = 0$.

(d). A matrix $M \in M_{n\times n}(F)$ has rank n if and only if $\det(M) \neq 0$.

(e). For any $A \in M_{n\times n}(F)$, $\det(A^t) = -\det(A)$.

(f). The determinant of a square matrix can be evaluated by cofactor expansion along any column.

(g). Every system of n linear equations in n unknowns can be solved by Cramer's rule.

(h). Let $A = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$. If $\det(A) \neq 0$ and if M_k is the $n \times n$ matrix obtained from A by replacing row k of A by b^t, then the unique solution of $Ax = b$ is

$$x_k = \frac{\det(M_k)}{\det(A)}, \qquad k = 1, 2, \ldots, n.$$

Xingjian's Solution:

(a). **False.** An elementary matrix can come from multiplying one row by a scalar $c \neq 0$, and then its determinant is c, not necessarily ± 1.

(b). **True.** This is the multiplicative property of determinants: $\det(AB) = \det(A)\det(B)$.

(c). **False.** A matrix is invertible if and only if its determinant is *nonzero*, not zero.

(d). **True.** For an $n\times n$ matrix, rank n means full rank, which is equivalent to invertibility, and hence equivalent to $\det(M) \neq 0$.

(e). **False.** The correct identity is $\det(A^t) = \det(A)$, not $-\det(A)$.

(f). **True.** A determinant may be computed by cofactor expansion along any row or any column.

(g). **False.** Cramer's rule applies only when the coefficient matrix has nonzero determinant.

(h). **False.** In Cramer's rule, one replaces the *column* k of A by b, not the row k. Therefore the statement is incorrect as written.

Question 2: Use Cramer's rule to solve the system

$$a_{11}x_1 + a_{12}x_2 = b_1, \qquad a_{21}x_1 + a_{22}x_2 = b_2,$$

where $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Xingjian's Solution:

The coefficient matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Its determinant is

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

For x_1, replace the first column of A by b:

$$A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}.$$

Hence

$$\det(A_1) = b_1 a_{22} - a_{12} b_2.$$

Therefore

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}.$$

For x_2, replace the second column of A by b:

$$A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}.$$

Hence

$$\det(A_2) = a_{11} b_2 - b_1 a_{21}.$$

Therefore

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}.$$

Thus the solution is

$$\boxed{x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}, \qquad x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}.}$$

Question 3: Use Cramer's rule to solve

$$2x_1 + x_2 - 3x_3 = 5,$$

$$x_1 - 2x_2 + x_3 = 10,$$

$$3x_1 + 4x_2 - 2x_3 = 0.$$

Xingjian's Solution:
The coefficient matrix and right-hand side vector are

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & 4 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}.$$

First compute

$$\det(A) = \begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & 4 & -2 \end{vmatrix}.$$

Expanding along the first row,

$$\det(A) = 2\begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}.$$

Thus

$$\det(A) = 2(4-4) - 1(-2-3) + (-3)(4+6) = 0 + 5 - 30 = -25.$$

Now compute A_1, A_2, A_3.

For x_1,

$$A_1 = \begin{bmatrix} 5 & 1 & -3 \\ 10 & -2 & 1 \\ 0 & 4 & -2 \end{bmatrix}.$$

Then

$$\det(A_1) = 5\begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} 10 & 1 \\ 0 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 10 & -2 \\ 0 & 4 \end{vmatrix}.$$

So

$$\det(A_1) = 5(4-4) - 1(-20) + (-3)(40) = 20 - 120 = -100.$$

Hence

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-100}{-25} = 4.$$

For x_2,

$$A_2 = \begin{bmatrix} 2 & 5 & -3 \\ 1 & 10 & 1 \\ 3 & 0 & -2 \end{bmatrix}.$$

Then

$$\det(A_2) = 2\begin{vmatrix} 10 & 1 \\ 0 & -2 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 10 \\ 3 & 0 \end{vmatrix}.$$

Thus

$$\det(A_2) = 2(-20) - 5(-5) + (-3)(-30) = -40 + 25 + 90 = 75.$$

Hence

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{75}{-25} = -3.$$

For x_3,

$$A_3 = \begin{bmatrix} 2 & 1 & 5 \\ 1 & -2 & 10 \\ 3 & 4 & 0 \end{bmatrix}.$$

Then

$$\det(A_3) = 2\begin{vmatrix} -2 & 10 \\ 4 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 10 \\ 3 & 0 \end{vmatrix} + 5\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}.$$

So

$$\det(A_3) = 2(0-40) - 1(0-30) + 5(4+6) = -80 + 30 + 50 = 0.$$

Hence

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{0}{-25} = 0.$$

Therefore,

$$\boxed{x_1 = 4, \qquad x_2 = -3, \qquad x_3 = 0.}$$

Question 4: Use Cramer's rule to solve

$$2x_1 + x_2 - 3x_3 = 1,$$

$$x_1 - 2x_2 + x_3 = 0,$$

$$3x_1 + 4x_2 - 2x_3 = -5.$$

Xingjian's Solution:

The coefficient matrix is the same as in Question 3:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & 4 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}.$$

We already know

$$\det(A) = -25.$$

For x_1,

$$A_1 = \begin{bmatrix} 1 & 1 & -3 \\ 0 & -2 & 1 \\ -5 & 4 & -2 \end{bmatrix}.$$

Then

$$\det(A_1) = 1\begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ -5 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 0 & -2 \\ -5 & 4 \end{vmatrix}.$$

Hence

$$\det(A_1) = 1(4-4) - 1(0-(-5)) + (-3)(0-10) = -5 + 30 = 25.$$

So

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{25}{-25} = -1.$$

For x_2,

$$A_2 = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & -5 & -2 \end{bmatrix}.$$

Then

$$\det(A_2) = 2\begin{vmatrix} 0 & 1 \\ -5 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 0 \\ 3 & -5 \end{vmatrix}.$$

Thus

$$\det(A_2) = 2(5) - 1(-5) + (-3)(-5) = 10 + 5 + 15 = 30.$$

So

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{30}{-25} = -\frac{6}{5}.$$

For x_3,

$$A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & 0 \\ 3 & 4 & -5 \end{bmatrix}.$$

Then

$$\det(A_3) = 2\begin{vmatrix} -2 & 0 \\ 4 & -5 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 3 & -5 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}.$$

Hence

$$\det(A_3) = 2(10) - 1(-5) + 1(4+6) = 20 + 5 + 10 = 35.$$

So

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{35}{-25} = -\frac{7}{5}.$$

Therefore,

$$\boxed{x_1 = -1, \qquad x_2 = -\frac{6}{5}, \qquad x_3 = -\frac{7}{5}.}$$

Question 5: Use Cramer's rule to solve

$$x_1 - x_2 + 4x_3 = -4,$$

$$-8x_1 + 3x_2 + x_3 = 8,$$

$$2x_1 - x_2 + x_3 = 0.$$

Xingjian's Solution:

Here

$$A = \begin{bmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix}.$$

Compute

$$\det(A) = \begin{vmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & -1 & 1 \end{vmatrix}.$$

Expanding along the first row,

$$\det(A) = 1\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -8 & 1 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} -8 & 3 \\ 2 & -1 \end{vmatrix}.$$

Thus

$$\det(A) = 1(3+1) + 1(-8-2) + 4(8-6) = 4 - 10 + 8 = 2.$$

For x_1,

$$A_1 = \begin{bmatrix} -4 & -1 & 4 \\ 8 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then

$$\det(A_1) = -4\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 8 & 1 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 8 & 3 \\ 0 & -1 \end{vmatrix}.$$

So

$$\det(A_1) = -4(4) + 1(8) + 4(-8) = -16 + 8 - 32 = -40.$$

Hence

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{2} = -20.$$

For x_2,

$$A_2 = \begin{bmatrix} 1 & -4 & 4 \\ -8 & 8 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Then

$$\det(A_2) = 1\begin{vmatrix} 8 & 1 \\ 0 & 1 \end{vmatrix} - (-4)\begin{vmatrix} -8 & 1 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} -8 & 8 \\ 2 & 0 \end{vmatrix}.$$

Hence

$$\det(A_2) = 8 + 4(-10) + 4(-16) = 8 - 40 - 64 = -96.$$

So

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-96}{2} = -48.$$

For x_3,

$$A_3 = \begin{bmatrix} 1 & -1 & -4 \\ -8 & 3 & 8 \\ 2 & -1 & 0 \end{bmatrix}.$$

Then

$$\det(A_3) = 1\begin{vmatrix} 3 & 8 \\ -1 & 0 \end{vmatrix} - (-1)\begin{vmatrix} -8 & 8 \\ 2 & 0 \end{vmatrix} + (-4)\begin{vmatrix} -8 & 3 \\ 2 & -1 \end{vmatrix}.$$

Thus

$$\det(A_3) = 8 + 16 + (-4)(8 - 6) = 8 + 16 - 8 = 16.$$

So

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{16}{2} = 8.$$

Therefore,

$$\boxed{x_1 = -20, \qquad x_2 = -48, \qquad x_3 = 8.}$$

Question 6: Use Cramer's rule to solve

$$x_1 - x_2 + 4x_3 = -2,$$

$$-8x_1 + 3x_2 + x_3 = 0,$$
$$2x_1 - x_2 + x_3 = 6.$$

Xingjian's Solution:

Again,

$$A = \begin{bmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}.$$

From Question 5,

$$\det(A) = 2.$$

For x_1,

$$A_1 = \begin{bmatrix} -2 & -1 & 4 \\ 0 & 3 & 1 \\ 6 & -1 & 1 \end{bmatrix}.$$

Then

$$\det(A_1) = -2\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 1 \\ 6 & 1 \end{vmatrix} + 4\begin{vmatrix} 0 & 3 \\ 6 & -1 \end{vmatrix}.$$

Hence

$$\det(A_1) = -2(4) + 1(0-6) + 4(0-18) = -8 - 6 - 72 = -86.$$

So

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-86}{2} = -43.$$

For x_2,

$$A_2 = \begin{bmatrix} 1 & -2 & 4 \\ -8 & 0 & 1 \\ 2 & 6 & 1 \end{bmatrix}.$$

Then

$$\det(A_2) = 1\begin{vmatrix} 0 & 1 \\ 6 & 1 \end{vmatrix} - (-2)\begin{vmatrix} -8 & 1 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} -8 & 0 \\ 2 & 6 \end{vmatrix}.$$

Thus

$$\det(A_2) = -6 + 2(-10) + 4(-48) = -6 - 20 - 192 = -218.$$

So

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-218}{2} = -109.$$

For x_3,

$$A_3 = \begin{bmatrix} 1 & -1 & -2 \\ -8 & 3 & 0 \\ 2 & -1 & 6 \end{bmatrix}.$$

Then

$$\det(A_3) = 1\begin{vmatrix} 3 & 0 \\ -1 & 6 \end{vmatrix} - (-1)\begin{vmatrix} -8 & 0 \\ 2 & 6 \end{vmatrix} + (-2)\begin{vmatrix} -8 & 3 \\ 2 & -1 \end{vmatrix}.$$

Hence

$$\det(A_3) = 18 - 48 + (-2)(8-6) = 18 - 48 - 4 = -34.$$

So

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-34}{2} = -17.$$

Therefore,

$$\boxed{x_1 = -43, \qquad x_2 = -109, \qquad x_3 = -17.}$$

Question 7: Use Cramer's rule to solve

$$3x_1 + x_2 + x_3 = 4,$$

$$-2x_1 - x_2 = 12,$$

$$x_1 + 2x_2 + x_3 = -8.$$

Xingjian's Solution:

Here

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 12 \\ -8 \end{bmatrix}.$$

Compute

$$\det(A) = \begin{vmatrix} 3 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix}.$$

Expanding along the first row,

$$\det(A) = 3\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix}.$$

Thus

$$\det(A) = 3(-1) - 1(-2) + 1(-4+1) = -3 + 2 - 3 = -4.$$

For x_1,

$$A_1 = \begin{bmatrix} 4 & 1 & 1 \\ 12 & -1 & 0 \\ -8 & 2 & 1 \end{bmatrix}.$$

Then

$$\det(A_1) = 4\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 12 & 0 \\ -8 & 1 \end{vmatrix} + 1\begin{vmatrix} 12 & -1 \\ -8 & 2 \end{vmatrix}.$$

Hence

$$\det(A_1) = 4(-1) - 12 + (24 - 8) = -4 - 12 + 16 = 0.$$

So

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{0}{-4} = 0.$$

For x_2,

$$A_2 = \begin{bmatrix} 3 & 4 & 1 \\ -2 & 12 & 0 \\ 1 & -8 & 1 \end{bmatrix}.$$

Then

$$\det(A_2) = 3\begin{vmatrix} 12 & 0 \\ -8 & 1 \end{vmatrix} - 4\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} -2 & 12 \\ 1 & -8 \end{vmatrix}.$$

Thus

$$\det(A_2) = 36 - 4(-2) + (16 - 12) = 36 + 8 + 4 = 48.$$

So

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{48}{-4} = -12.$$

For x_3,

$$A_3 = \begin{bmatrix} 3 & 1 & 4 \\ -2 & -1 & 12 \\ 1 & 2 & -8 \end{bmatrix}.$$

Then

$$\det(A_3) = 3\begin{vmatrix} -1 & 12 \\ 2 & -8 \end{vmatrix} - 1\begin{vmatrix} -2 & 12 \\ 1 & -8 \end{vmatrix} + 4\begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix}.$$

Hence

$$\det(A_3) = 3(8 - 24) - 1(16 - 12) + 4(-4 + 1) = -48 - 4 - 12 = -64.$$

So

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-64}{-4} = 16.$$

Therefore,

$$\boxed{x_1 = 0, \qquad x_2 = -12, \qquad x_3 = 16.}$$

Question 9: Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.

Xingjian's Solution:

Let

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

be an upper triangular matrix.

For an upper triangular matrix, the determinant is the product of the diagonal entries:

$$\det(U) = u_{11}u_{22}\cdots u_{nn}.$$

Now U is invertible if and only if $\det(U) \neq 0$. Therefore,

$$U \text{ is invertible } \iff u_{11}u_{22}\cdots u_{nn} \neq 0.$$

A product of scalars is nonzero if and only if each factor is nonzero. Hence

$$U \text{ is invertible } \iff u_{11} \neq 0,\ u_{22} \neq 0,\ \ldots,\ u_{nn} \neq 0.$$

Therefore, an upper triangular $n \times n$ matrix is invertible if and only if all of its diagonal entries are nonzero. □

Question 10: A matrix $M \in M_{n\times n}(\mathbb{C})$ is called nilpotent if, for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that if M is nilpotent, then $\det(M) = 0$.

Xingjian's Solution:

Assume that M is nilpotent. Then there exists a positive integer k such that $M^k = O$.

Taking determinants on both sides gives $\det(M^k) = \det(O)$.

Using the multiplicative property of determinants,

$$\det(M^k) = \big(\det(M)\big)^k.$$

Also, $\det(O) = 0$. Hence $\big(\det(M)\big)^k = 0$. Since the only scalar whose kth power is 0 is 0 itself, it follows that $\det(M) = 0$.

Therefore every nilpotent matrix has determinant zero. □

Question 12: A matrix $Q \in M_{n\times n}(\mathbb{R})$ is called **orthogonal** if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

Xingjian's Solution:

Suppose Q is orthogonal, so $QQ^t = I$. Taking determinants on both sides gives $\det(QQ^t) = \det(I)$.

Using the product rule for determinants, $\det(Q)\det(Q^t) = 1$.

Since the determinant of a transpose equals the determinant of the matrix, $\det(Q^t) = \det(Q)$.

Therefore, $\det(Q)^2 = 1$. Hence $\det(Q) = \pm 1$.

Thus every orthogonal matrix has determinant either 1 or -1. □

Question 14: Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of F^n containing n distinct vectors, and let $B \in M_{n\times n}(F)$ be the matrix having u_j as column j. Prove that β is a basis for F^n if and only if $\det(B) \neq 0$.

Xingjian's Solution:

Proof. Let

$$B = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{pmatrix}.$$

The set β is a basis for F^n if and only if the vectors $\{u_1, \ldots, u_n\}$ are linearly independent.

A standard result states that a square matrix is invertible if and only if its columns are linearly independent, and this is equivalent to its determinant being nonzero. Therefore,

$$\beta \text{ is a basis for } F^n \iff B \text{ is invertible} \iff \det(B) \neq 0.$$

□

Question 15: Prove that if $A, B \in M_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.
Xingjian's Solution:

Proof. Since A and B are similar, there exists an invertible matrix P such that $B = P^{-1}AP$. Taking determinants,

$$\det(B) = \det(P^{-1}AP).$$

Using multiplicativity,

$$\det(B) = \det(P^{-1})\det(A)\det(P).$$

Since $\det(P^{-1}) = \frac{1}{\det(P)}$, we obtain

$$\det(B) = \det(A).$$

□

Question 17: Let $A, B \in M_{n\times n}(F)$ be such that $AB = -BA$. Prove that if n is odd and F is not of characteristic two, then A or B is not invertible.
Xingjian's Solution:

Proof. Assume both A and B are invertible. Taking determinants of $AB = -BA$, we get

$$\det(AB) = \det(-BA).$$

Using properties of determinants,

$$\det(AB) = \det(A)\det(B),$$

and

$$\det(-BA) = \det(-I)\det(BA) = (-1)^n \det(B)\det(A).$$

Thus,

$$\det(A)\det(B) = (-1)^n \det(A)\det(B).$$

Since n is odd, $(-1)^n = -1$, so

$$\det(A)\det(B) = -\det(A)\det(B).$$

Hence,

$$2\det(A)\det(B) = 0.$$

Since the field is not of characteristic two, $2 \neq 0$, so

$$\det(A)\det(B) = 0.$$

Thus at least one of $\det(A)$ or $\det(B)$ is zero, so at least one of A or B is not invertible. □

Question 20: Suppose that $M \in M_{n\times n}(F)$ can be written as

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix},$$

where A is a square matrix. Prove that $\det(M) = \det(A)$.
Xingjian's Solution:

Proof. The matrix M is block upper triangular. Its diagonal entries consist of the diagonal entries of A together with those of the identity matrix I.

Thus,

$$\det(M) = \det(A) \cdot \det(I).$$

Since $\det(I) = 1$, we conclude

$$\det(M) = \det(A).$$

□

Question 21: Suppose that $M \in M_{n\times n}(F)$ can be written as

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A and C are square matrices. Prove that $\det(M) = \det(A)\det(C)$.

Xingjian's Solution:

Proof. The matrix M is block upper triangular. Its diagonal entries consist of those of A and C. Therefore,

$$\det(M) = \det(A)\det(C).$$

□

Xingjian Xu
xingjianxu@ufl.edu
LIT 453
Webpage: https://people.clas.ufl.edu/xingjianxu/mas4105-26springs/
Office Hours: Tuesday, Periods 3 and 5 on LIT453;