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Question 1: Label the following statements as true or false.

(a). The function $\det : M_{n\times n}(F) \to F$ is a linear transformation.

(b). The determinant of a square matrix can be evaluated by cofactor expansion along any row.

(c). If two rows of a square matrix A are identical, then $\det(A) = 0$.

(d). If B is a matrix obtained from a square matrix A by interchanging any two rows, then $\det(B) = -\det(A)$.

(e). If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.

(f). If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = k\det(A)$.

(g). If $A \in M_{n\times n}(F)$ has rank n, then $\det(A) = 0$.

(h). The determinant of an upper triangular matrix equals the product of its diagonal entries.

Xingjian's Solution:

(a). **False.** The determinant is not linear as a map from $M_{n\times n}(F) \to F$.

(b). **True.** By the cofactor expansion theorem, we may expand along any row.

(c). **True.** If two rows are identical, then the determinant is zero.

(d). **True.** Interchanging two rows changes the sign of the determinant.

(e). **False.** If one row is multiplied by a scalar λ, then the determinant is multiplied by λ.

(f). **False.** Adding k times one row to another row does not change the determinant.

(g). **False.** If $\text{rank}(A) = n$, then A is invertible, so $\det(A) \neq 0$.

(h). **True.** The determinant of an upper triangular matrix is the product of its diagonal entries.

Question 2: Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Xingjian's Solution:

$$\det\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix}$$

is obtained by multiplying each row of

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

by 3. Since the determinant is linear in each row, we get

$$\det\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = 3\cdot 3\cdot 3\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 27\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Hence, $k = 27$.

Question 3: Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Xingjian's Solution: Let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Then the left-hand side is

$$\det\begin{pmatrix} 2R_1 \\ 3R_2+5R_3 \\ 7R_3 \end{pmatrix},$$

where R_1, R_2, R_3 are the rows of A. Using linearity of the determinant in each row,

$$\det\begin{pmatrix} 2R_1 \\ 3R_2+5R_3 \\ 7R_3 \end{pmatrix} = 2\cdot 7\det\begin{pmatrix} R_1 \\ 3R_2+5R_3 \\ R_3 \end{pmatrix}.$$

Expanding along the second row,

$$= 14\left(3\det\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} + 5\det\begin{pmatrix} R_1 \\ R_3 \\ R_3 \end{pmatrix}\right).$$

Since a determinant with two identical rows is zero,

$$\det\begin{pmatrix} R_1 \\ R_3 \\ R_3 \end{pmatrix} = 0.$$

Therefore,

$$\det\begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{pmatrix} = 14\cdot 3\det(A) = 42\det(A).$$

Hence, $k = 42$.

Question 4: Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Xingjian's Solution: Let

$$R_1 = (a_1, a_2, a_3), \qquad R_2 = (b_1, b_2, b_3), \qquad R_3 = (c_1, c_2, c_3).$$

Then the determinant on the left-hand side is

$$\det\begin{pmatrix} R_2+R_3 \\ R_1+R_3 \\ R_1+R_2 \end{pmatrix}.$$

Using linearity in the rows,

$$\det\begin{pmatrix} R_2+R_3 \\ R_1+R_3 \\ R_1+R_2 \end{pmatrix} = \det\begin{pmatrix} R_2 \\ R_1+R_3 \\ R_1+R_2 \end{pmatrix} + \det\begin{pmatrix} R_3 \\ R_1+R_3 \\ R_1+R_2 \end{pmatrix}.$$

Expanding completely, every term with two equal rows is zero, so only

$$\det\begin{pmatrix} R_2 \\ R_3 \\ R_1 \end{pmatrix} + \det\begin{pmatrix} R_3 \\ R_1 \\ R_2 \end{pmatrix}$$

remains. Each of these is a cyclic permutation of

$$\det\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix},$$

hence each has the same determinant. Therefore,

$$\det\begin{pmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = 2\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Hence, $k = 2$.

Question 7: Evaluate the determinant of the following matrix by cofactor expansion along the second row:

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}.$$

Xingjian's Solution:

By cofactor expansion along the second row, we get

$$\det(A) = (-1)C_{21} + 0 \cdot C_{22} + (-3)C_{23}.$$

Now

$$C_{21} = (-1)^{2+1} \det \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = -(1 \cdot 0 - 2 \cdot 3) = 6,$$

and

$$C_{23} = (-1)^{2+3} \det \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = -(0 \cdot 3 - 1 \cdot 2) = 2.$$

Therefore,

$$\det(A) = (-1)(6) + (-3)(2) = -6 - 6 = -12.$$

Question 11: Evaluate the determinant of the following matrix by cofactor expansion along the fourth row:

$$\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}.$$

Xingjian's Solution:

By cofactor expansion along the fourth row, we get

$$\det(A) = (-1)C_{41} + 1 \cdot C_{42} + 2C_{43} + 0 \cdot C_{44}.$$

First,

$$C_{41} = (-1)^{4+1} \det \begin{pmatrix} 2 & 1 & 3 \\ 0 & -2 & 2 \\ -1 & 0 & 1 \end{pmatrix} = -\det \begin{pmatrix} 2 & 1 & 3 \\ 0 & -2 & 2 \\ -1 & 0 & 1 \end{pmatrix}.$$

Compute

$$\det \begin{pmatrix} 2 & 1 & 3 \\ 0 & -2 & 2 \\ -1 & 0 & 1 \end{pmatrix} = 2 \begin{vmatrix} -2 & 2 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 0 & -2 \\ -1 & 0 \end{vmatrix} = 2(-2) - 1(2) + 3(-2) = -12.$$

Hence

$$C_{41} = 12.$$

Next,

$$C_{42} = (-1)^{4+2} \det \begin{pmatrix} 0 & 1 & 3 \\ 1 & -2 & 2 \\ 3 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 3 \\ 1 & -2 & 2 \\ 3 & 0 & 1 \end{pmatrix}.$$

Compute

$$\det \begin{pmatrix} 0 & 1 & 3 \\ 1 & -2 & 2 \\ 3 & 0 & 1 \end{pmatrix} = 0 - 1 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix} = -(1 - 6) + 3(0 + 6) = 23.$$

So $C_{42} = 23$.

Next,

$$C_{43} = (-1)^{4+3} \det \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix} = - \det \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix}.$$

Compute

$$\det \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix} = 0 - 2 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} = -2(1-6) + 3(-1) = 7.$$

Hence $C_{43} = -7$.

Therefore,

$$\det(A) = (-1)(12) + 1(23) + 2(-7) = -12 + 23 - 14 = -3.$$

Question 12: Evaluate the determinant of the following matrix by cofactor expansion along the fourth row:

$$\begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}.$$

Xingjian's Solution:

By cofactor expansion along the fourth row, we get

$$\det(A) = (-2)C_{41} + 6C_{42} + (-4)C_{43} + 1 \cdot C_{44}.$$

First,

$$C_{41} = (-1)^{4+1} \det \begin{pmatrix} -1 & 2 & -1 \\ 4 & 1 & -1 \\ -5 & -3 & 8 \end{pmatrix} = - \det \begin{pmatrix} -1 & 2 & -1 \\ 4 & 1 & -1 \\ -5 & -3 & 8 \end{pmatrix}.$$

The 3×3 determinant is

$$-1 \begin{vmatrix} 1 & -1 \\ -3 & 8 \end{vmatrix} - 2 \begin{vmatrix} 4 & -1 \\ -5 & 8 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 1 \\ -5 & -3 \end{vmatrix} = -5 - 54 + 7 = -52.$$

Thus $C_{41} = 52$.

Next,

$$C_{42} = (-1)^{4+2} \det \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & -1 \\ 2 & -3 & 8 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & -1 \\ 2 & -3 & 8 \end{pmatrix}.$$

Compute

$$1 \begin{vmatrix} 1 & -1 \\ -3 & 8 \end{vmatrix} - 2 \begin{vmatrix} -3 & -1 \\ 2 & 8 \end{vmatrix} + (-1) \begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix} = 5 + 44 - 7 = 42.$$

Hence $C_{42} = 42$.

Next,

$$C_{43} = (-1)^{4+3} \det \begin{pmatrix} 1 & -1 & -1 \\ -3 & 4 & -1 \\ 2 & -5 & 8 \end{pmatrix} = - \det \begin{pmatrix} 1 & -1 & -1 \\ -3 & 4 & -1 \\ 2 & -5 & 8 \end{pmatrix}.$$

Compute

$$1 \begin{vmatrix} 4 & -1 \\ -5 & 8 \end{vmatrix} - (-1) \begin{vmatrix} -3 & -1 \\ 2 & 8 \end{vmatrix} + (-1) \begin{vmatrix} -3 & 4 \\ 2 & -5 \end{vmatrix} = 27 - 22 - 7 = -2.$$

Therefore $C_{43} = 2$.

Finally,

$$C_{44} = (-1)^{4+4} \det \begin{pmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 2 & -5 & -3 \end{pmatrix} = \det \begin{pmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 2 & -5 & -3 \end{pmatrix}.$$

Compute

$$1 \begin{vmatrix} 4 & 1 \\ -5 & -3 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix} + 2 \begin{vmatrix} -3 & 4 \\ 2 & -5 \end{vmatrix} = -7 + 7 + 14 = 14.$$

Hence $C_{44} = 14$.

Therefore,

$$\det(A) = (-2)(52) + 6(42) + (-4)(2) + 14 = -104 + 252 - 8 + 14 = 154.$$

Question 23: Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Xingjian's Solution: Let $A = (a_{ij}) \in M_{n\times n}(F)$ be an upper triangular matrix. Then $a_{ij} = 0$ whenever $i > j$. We prove that

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

by induction on n.

If $n = 1$, then the statement is obvious.

Assume the formula is true for every $(n-1)\times(n-1)$ upper triangular matrix. Now let A be an $n \times n$ upper triangular matrix. Since all entries below a_{11} in the first column are zero, cofactor expansion along the first column gives

$$\det(A) = a_{11} \det(A_{11}),$$

where A_{11} is the matrix obtained by deleting the first row and first column. Since A_{11} is again upper triangular, the induction hypothesis implies

$$\det(A_{11}) = a_{22}a_{33}\cdots a_{nn}.$$

Therefore, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

Question 24: Prove the corollary to Theorem 4.3.

If $A \in M_{n\times n}(F)$ has a row consisting entirely of zeros, then $\det(A) = 0$

Xingjian's Solution:
Let

$$A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \in M_{n\times n}(F),$$

and suppose that one row of A is the zero row. Without loss of generality, assume $\mathbf{r}_k = \mathbf{0}$ for some $k \in \{1, \ldots, n\}$.

Since the determinant is linear in each row, if all rows except the kth are fixed, then

$$\det(A) = \det(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, \mathbf{0}, \mathbf{r}_{k+1}, \ldots, \mathbf{r}_n).$$

But

$$\mathbf{0} = 0\mathbf{v}$$

for any row vector $\mathbf{v}$, so by linearity in the kth row,

$$\det(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, \mathbf{0}, \mathbf{r}_{k+1}, \ldots, \mathbf{r}_n) = \det(\mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, 0\mathbf{v}, \mathbf{r}_{k+1}, \ldots, \mathbf{r}_n) = 0.$$

Therefore,

$$\boxed{\det(A) = 0.}$$

□

Question 25: Prove that $\det(kA) = k^n \det(A)$ for any $A \in M_{n\times n}(F)$.
Xingjian's Solution:
Let $A \in M_{n\times n}(F)$. Then kA is obtained by multiplying every row of A by k. Since the determinant is linear in each row, we may factor out k once from each of the n rows. Thus

$$\det(kA) = k \cdot k \cdots k \det(A) = k^n \det(A).$$

Question 28: Compute $\det(E_i)$ if E_i is an elementary matrix of type i.
Xingjian's Solution:
There are three types of elementary matrices.

- If E_i is of type 1, then it is obtained by interchanging two rows of the identity matrix. Hence
$$\det(E_i) = -1.$$
- If E_i is of type 2, then it is obtained by multiplying one row of the identity matrix by a nonzero scalar c. Hence
$$\det(E_i) = c.$$
- If E_i is of type 3, then it is obtained by adding a multiple of one row of the identity matrix to another row. This operation does not change the determinant, so
$$\det(E_i) = 1.$$

Question 29: Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.
Xingjian's Solution:
Let E be an elementary matrix. We want to prove that

$$\det(E^t) = \det(E).$$

We consider the three types of elementary matrices.

- If E is of type 1, then $E^t = E$. Hence

$$\det(E^t) = \det(E) = -1.$$

- If E is of type 2, then $E^t = E$. Hence

$$\det(E^t) = \det(E).$$

- If E is of type 3, then E^t is again an elementary matrix of type 3. Since a type 3 elementary matrix has determinant 1, we have

$$\det(E^t) = 1 = \det(E).$$

Therefore, in all cases,

$$\det(E^t) = \det(E).$$

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