Question 1: Label the following statements as true or false.

(a). The function $\det : M_{2\times 2}(F) \to F$ is a linear transformation.

(b). The determinant of a 2×2 matrix is a linear function of each row of the matrix when the other row is held fixed.

(c). If $A \in M_{2\times 2}(F)$ and $\det(A) = 0$, then A is invertible.

(d). If u and v are vectors in $\mathbb{R}^2$ emanating from the origin, then the area of the parallelogram having u and v as adjacent sides is $\det \begin{pmatrix} u \\ v \end{pmatrix}$.

(e). A coordinate system is right-handed if and only if its orientation equals 1.

Xingjian's Solution:

(a). **False.** The determinant is not linear in the whole matrix. In general, $\det(A+B) \neq \det(A) + \det(B)$.

(b). **True.** For a 2×2 matrix, the determinant is linear in each row separately when the other row is fixed.

(c). **False.** A matrix is invertible if and only if its determinant is nonzero. So if $\det(A) = 0$, then A is not invertible.

(d). **False.** The area of the parallelogram is $\left|\det \begin{pmatrix} u \\ v \end{pmatrix}\right|$, not necessarily the determinant itself.

(e). **True.** A coordinate system is right-handed precisely when its orientation is 1.

Question 2: Compute the determinants of the following matrices in $M_{2\times 2}(\mathbb{R})$.

(a). $\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$ (b). $\begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix}$ (c). $\begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix}$

Xingjian's Solution:

For a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

(a). $\det \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix} = 6 \cdot 4 - (-3) \cdot 2 = 24 + 6 = 30.$

(b). $\det \begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix} = (-5)(1) - 2 \cdot 6 = -5 - 12 = -17.$

(c). $\det \begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix} = 8(-1) - 0 \cdot 3 = -8.$

Question 3: Compute the determinants of the following matrices in $M_{2\times 2}(\mathbb{C})$.

(a). $\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix}$ (b). $\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix}$ (c). $\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix}$

Xingjian's Solution:

Again, for a 2×2 matrix,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

(a). $\det \begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix} = (-1+i)(2-3i) - (1-4i)(3+2i)$. First,

$$(-1+i)(2-3i) = -2 + 3i + 2i - 3i^2 = 1 + 5i,$$

since $i^2 = -1$. Next,

$$(1-4i)(3+2i) = 3+2i-12i-8i^2 = 11-10i.$$

Hence

$$\det = (1+5i) - (11-10i) = -10+15i.$$

(b). $\det \begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix} = (5-2i)(7i) - (6+4i)(-3+i)$. Compute

$$(5-2i)(7i) = 35i - 14i^2 = 14+35i,$$

and

$$(6+4i)(-3+i) = -18+6i-12i+4i^2 = -22-6i.$$

Therefore

$$\det = (14+35i) - (-22-6i) = 36+41i.$$

(c). $\det \begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix} = (2i)(6i) - 3\cdot 4 = 12i^2 - 12 = -12-12 = -24.$

Question 4: For each of the following pairs of vectors u and v in $\mathbb{R}^2$, compute the area of the parallelogram determined by u and v.

(a). $u = (3,-2)$ and $v = (2,5)$

(b). $u = (1,3)$ and $v = (-3,1)$

(c). $u = (4,-1)$ and $v = (-6,-2)$

(d). $u = (3,4)$ and $v = (2,-6)$

Xingjian's Solution:

The area of the parallelogram determined by $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is

$$\left| \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right|.$$

(a). Area $= \left| \det \begin{pmatrix} 3 & -2 \\ 2 & 5 \end{pmatrix} \right| = |3\cdot 5 - (-2)\cdot 2| = |15+4| = 19.$

(b). Area $= \left| \det \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \right| = |1\cdot 1 - 3(-3)| = |1+9| = 10.$

(c). Area $= \left| \det \begin{pmatrix} 4 & -1 \\ -6 & -2 \end{pmatrix} \right| = |4(-2) - (-1)(-6)| = |-8-6| = 14.$

(d). Area $= \left| \det \begin{pmatrix} 3 & 4 \\ 2 & -6 \end{pmatrix} \right| = |3(-6) - 4\cdot 2| = |-18-8| = 26.$

Question 8: Prove that if $A \in M_{2\times 2}(F)$ is upper triangular, then $\det(A)$ equals the product of the diagonal entries of A.

Xingjian's Solution:

Let

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

be an upper triangular matrix in $M_{2\times 2}(F)$. By the definition of the determinant of a 2×2 matrix,

$$\det(A) = ad - b \cdot 0 = ad.$$

Thus $\det(A)$ is exactly the product of the diagonal entries of A.

Therefore,

$$\boxed{\det(A) = ad.}$$

Question 9: Prove that $\det(AB) = \det(A)\det(B)$ for any $A, B \in M_{2\times 2}(F)$.

Xingjian's Solution:

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Hence

$$\det(AB) = (ae + bg)(cf + dh) - (af + bh)(ce + dg).$$

Now expand:

$$\det(AB) = aecf + aedh + bgcf + bgdh - afce - afdg - bhce - bhdg.$$

Rearranging terms,

$$\det(AB) = (aedh - afdg) + (bgcf - bhce) + (aecf - afce) + (bgdh - bhdg).$$

Notice that

$$aecf - afce = 0, \qquad bgdh - bhdg = 0,$$

so

$$\det(AB) = aedh - afdg + bgcf - bhce.$$

Factor this as

$$\det(AB) = ad(eh - fg) - bc(eh - fg).$$

Therefore,

$$\det(AB) = (ad - bc)(eh - fg).$$

Since

$$\det(A) = ad - bc, \qquad \det(B) = eh - fg,$$

we conclude that

$$\boxed{\det(AB) = \det(A)\det(B).}$$

Question 10: The classical adjoint of a 2×2 matrix $A \in M_{2\times 2}(F)$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Prove that

(a). $CA = AC = [\det(A)]I$.

(b). $\det(C) = \det(A)$.

(c). The classical adjoint of A^t is C^t.

(d). If A is invertible, then $A^{-1} = [\det(A)]^{-1}C$.

Xingjian's Solution:

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then its classical adjoint is

$$C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(a) We compute

$$CA = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} da - bc & db - bd \\ -ac + ac & -cb + ad \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}.$$

Thus

$$CA = (ad - bc)I = [\det(A)]I.$$

Similarly,

$$AC = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & -ab + ab \\ cd - dc & -cb + ad \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}.$$

Hence

$$AC = [\det(A)]I.$$

Therefore,

$$\boxed{CA = AC = [\det(A)]I.}$$

(b) Since

$$C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

we have

$$\det(C) = da - (-b)(-c) = ad - bc = \det(A).$$

Therefore,

$$\boxed{\det(C) = \det(A).}$$

(c) Since

$$A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

the classical adjoint of A^t is

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

On the other hand,

$$C^t = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Thus the classical adjoint of A^t is exactly C^t.

Therefore,

$$\boxed{\text{adj}(A^t) = C^t.}$$

(d) If A is invertible, then $\det(A) \neq 0$. From part (a),

$$AC = [\det(A)]I.$$

Multiply both sides by $[\det(A)]^{-1}$:

$$A\left([\det(A)]^{-1}C\right) = I.$$

Similarly,

$$\left([\det(A)]^{-1}C\right)A = I.$$

Hence $[\det(A)]^{-1}C$ is the inverse of A. Therefore,

$$\boxed{A^{-1} = [\det(A)]^{-1}C.}$$

Question 12: Let $\{u, v\}$ be an ordered basis for $\mathbb{R}^2$. Prove that

$$O\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if $\{u, v\}$ forms a right-handed coordinate system.

Xingjian's Solution:

Let $u = (u_1, u_2), v = (v_1, v_2)$, and consider the matrix whose rows are u and v:

$$A = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}.$$

Its orientation is

$$O\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det(A)}{|\det(A)|}.$$

Since $\{u, v\}$ is an ordered basis for $\mathbb{R}^2$, the vectors u and v are linearly independent, so $\det(A) \neq 0$. Thus

$$O\begin{pmatrix} u \\ v \end{pmatrix} = 1 \quad \Longleftrightarrow \quad \det(A) > 0.$$

Now recall that in $\mathbb{R}^2$, an ordered basis $\{u, v\}$ forms a right-handed coordinate system exactly when the rotation from u to v is counterclockwise through an angle less than π. This happens precisely when the signed area determined by u and v is positive, that is,

$$\det(A) > 0.$$

Therefore,

$$\{u, v\} \text{ is right-handed} \iff \det(A) > 0 \iff O\begin{pmatrix} u \\ v \end{pmatrix} = 1.$$

Hence,

$$\boxed{O\begin{pmatrix} u \\ v \end{pmatrix} = 1 \iff \{u, v\} \text{ forms a right-handed coordinate system.}}$$

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