**Xingjian comment:Reduced Row Echelon Form (RREF):**
Leading 1 (pivot) in each nonzero row
Each pivot is the only nonzero entry in its column
Pivots move to the right as you go down
All zero rows are at the bottom

**Question 1:** Label the following statements as true or false.

(a). If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary column operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.

(b). If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary row operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.

(c). If $A$ is an $n \times n$ matrix with rank $n$, then the reduced row echelon form of $A$ is $I_n$.

(d). Any matrix can be put in reduced row echelon form by means of a finite sequence of elementary row operations.

(e). If $(A|b)$ is in reduced row echelon form, then the system $Ax = b$ is consistent.

(f). Let $Ax = b$ be a system of $m$ linear equations in $n$ unknowns for which the augmented matrix is in reduced row echelon form. If this system is consistent, then the dimension of the solution set of $Ax = 0$ is $n - r$, where $r$ equals the number of nonzero rows in $A$.

(g). If a matrix $A$ is transformed by elementary row operations into a matrix $A'$ in reduced row echelon form, then the number of nonzero rows in $A'$ equals the rank of $A$.

*Xingjian's Solution:*

(a) **False.** Elementary column operations change the coefficient matrix in a way that generally changes the variables, so the original system and the new one are not necessarily equivalent.

(b) **True.** Elementary row operations preserve the solution set of the system, so the two systems are equivalent.

(c) **True.** If $A$ is an $n \times n$ matrix with rank $n$, then $A$ has a pivot in every row and every column, so its reduced row echelon form must be $I_n$.

(d) **True.** Every matrix can be transformed into reduced row echelon form by a finite sequence of elementary row operations.

(e) **False.** An augmented matrix may be in reduced row echelon form and still be inconsistent, for example if it contains a row of the form

$$(0 \ \cdots \ 0 \,|\, 1).$$

(f) **True.** If the system is consistent and the augmented matrix is in reduced row echelon form, then $r$ is the number of pivot variables. Hence there are $n - r$ free variables, so the dimension of the solution set of $Ax = 0$ is $n - r$.

(g) **True.** Rank is invariant under elementary row operations, and in reduced row echelon form the rank equals the number of nonzero rows. Therefore the number of nonzero rows in $A'$ equals rank$(A)$.

**Question 2:** Use Gaussian elimination to solve the following systems of linear equations.

(a). $\begin{cases} x_1 + 2x_2 - x_3 = -1 \\ 2x_1 + 2x_2 + x_3 = 1 \\ 3x_1 + 5x_2 - 2x_3 = -1 \end{cases}$

*Xingjian's Solution:*

$$(A|b) = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right] \xrightarrow{R_2-2R_1,\, R_3-3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_3\cdot(-2)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Back-substitution gives: $x_3 = -1, \quad x_2 = 0, \quad x_1 = -2$

$$\boxed{(x_1, x_2, x_3) = (-2, 0, -1)}$$

(b). $\begin{cases} x_1 - 2x_2 - x_3 = 1 \\ 2x_1 - 3x_2 + x_3 = 6 \\ 3x_1 - 5x_2 = 7 \\ x_1 + 5x_3 = 9 \end{cases}$

*Xingjian's Solution:*

$$(A|b) = \left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 2 & -3 & 1 & 6 \\ 3 & -5 & 0 & 7 \\ 1 & 0 & 5 & 9 \end{array}\right] \xrightarrow{R_2-2R_1,\, R_3-3R_1,\, R_4-R_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 6 & 8 \end{array}\right] \xrightarrow{R_3-R_2,\, R_4-2R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the system has infinitely many solutions. Let $x_3 = t$. Then from the second row,

$$x_2 + 3x_3 = 4 \quad \Longrightarrow \quad x_2 = 4 - 3t,$$

and from the first row,

$$x_1 - 2x_2 - x_3 = 1 \quad \Longrightarrow \quad x_1 = 1 + 2(4 - 3t) + t = 9 - 5t.$$

Therefore,

$$\boxed{(x_1, x_2, x_3) = (9 - 5t,\ 4 - 3t,\ t), \qquad t \in \mathbb{R}.}$$

(c). $\begin{cases} x_1 + 2x_2 + 2x_4 = 6 \\ 3x_1 + 5x_2 - x_3 + 6x_4 = 17 \\ 2x_1 + 4x_2 + x_3 + 2x_4 = 12 \\ 2x_1 - 7x_3 + 11x_4 = 7 \end{cases}$

*Xingjian's Solution*:

$$(A|b) = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 3 & 5 & -1 & 6 & 17 \\ 2 & 4 & 1 & 2 & 12 \\ 2 & 0 & -7 & 11 & 7 \end{array}\right] \xrightarrow{R_2-3R_1,\ R_3-2R_1,\ R_4-2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & -4 & -7 & 7 & -5 \end{array}\right]$$

$$\xrightarrow{R_4-4R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -3 & 7 & -1 \end{array}\right] \xrightarrow{R_4+3R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Back-substitution gives

$$x_4 = -1, \qquad x_3 = -2, \qquad x_2 = 3, \qquad x_1 = 2.$$

$$\boxed{(x_1, x_2, x_3, x_4) = (2, 3, -2, -1)}$$

(d). $\begin{cases} x_1 - x_2 - 2x_3 + 3x_4 = -7 \\ 2x_1 - x_2 + 6x_3 + 6x_4 = -2 \\ -2x_1 + x_2 - 4x_3 - 3x_4 = 0 \\ 3x_1 - 2x_2 + 9x_3 + 10x_4 = -5 \end{cases}$

*Xingjian's Solution*:

$$(A|b) = \left[\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 2 & -1 & 6 & 6 & -2 \\ -2 & 1 & -4 & -3 & 0 \\ 3 & -2 & 9 & 10 & -5 \end{array}\right] \xrightarrow{R_2-2R_1,\ R_3+2R_1,\ R_4-3R_1} \left[\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & -1 & -8 & 3 & -14 \\ 0 & 1 & 15 & 1 & 16 \end{array}\right]$$

$$\xrightarrow{R_3+R_2,\ R_4-R_2} \left[\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 5 & 1 & 4 \end{array}\right] \xrightarrow{2R_4-5R_3} \left[\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & -13 & 18 \end{array}\right]$$

Hence $x_4 = -\frac{18}{13}$. From the third row,

$$2x_3 + 3x_4 = -2 \quad \Longrightarrow \quad 2x_3 - \frac{54}{13} = -2 \quad \Longrightarrow \quad x_3 = \frac{14}{13}.$$

From the second row,

$$x_2 + 10x_3 = 12 \quad \Longrightarrow \quad x_2 = 12 - \frac{140}{13} = \frac{16}{13}.$$

From the first row,

$$x_1 - x_2 - 2x_3 + 3x_4 = -7$$

so

$$x_1 = -7 + \frac{16}{13} + \frac{28}{13} + \frac{54}{13} = \frac{7}{13}.$$

Therefore,

$$\boxed{(x_1, x_2, x_3, x_4) = \left(\frac{7}{13}, \frac{16}{13}, \frac{14}{13}, -\frac{18}{13}\right).}$$

(e). $\begin{cases} x_1 - 4x_2 - x_3 + x_4 = 3 \\ 2x_1 - 8x_2 + x_3 - 4x_4 = 9 \\ -x_1 + 4x_2 - 2x_3 + 5x_4 = -6 \end{cases}$

*Xingjian's Solution*:

$$(A|b) = \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 2 & -8 & 1 & -4 & 9 \\ -1 & 4 & -2 & 5 & -6 \end{array}\right]$$

$$\xrightarrow{R_2-2R_1,\ R_3+R_1} \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & -3 & 6 & -3 \end{array}\right] \xrightarrow{\frac{1}{3}R_2,\ R_3+3R_2} \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_2 = s, x_4 = t$. Then from the second row, $x_3 - 2x_4 = 1 \implies x_3 = 1 + 2t$. From the first row,

$$x_1 - 4x_2 - x_3 + x_4 = 3$$

so

$$x_1 = 3 + 4s + x_3 - x_4 = 3 + 4s + (1 + 2t) - t = 4 + 4s + t.$$

Therefore,

$$\boxed{(x_1, x_2, x_3, x_4) = (4 + 4s + t,\ s,\ 1 + 2t,\ t), \qquad s, t \in \mathbb{R}.}$$

(f). $\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 = 5 \\ x_2 + 2x_4 = 3 \end{cases}$

*Xingjian's Solution*:

$$(A|b) = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Let $x_4 = t$. Then $x_3 = 1, \quad x_2 = 3 - 2t$. From the first row,

$$x_1 + 2x_2 - x_3 + 3x_4 = 2$$

so

$$x_1 = 2 - 2(3 - 2t) + 1 - 3t = t - 3.$$

Therefore,

$$\boxed{(x_1, x_2, x_3, x_4) = (t-3,\ 3-2t,\ 1,\ t), \qquad t \in \mathbb{R}.}$$

(g). $\begin{cases} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 = 1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 = 2 \\ 4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 = 6 \end{cases}$

*Xingjian's Solution*:

$$(A|b) = \left[\begin{array}{ccccc|c} 2 & -2 & -1 & 6 & -2 & 1 \\ 1 & -1 & 1 & 2 & -1 & 2 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 2 & -2 & -1 & 6 & -2 & 1 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1,\ R_3 - 4R_1} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & -3 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 & 3 & -2 \end{array}\right] \xrightarrow{R_2 + 3R_3} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & -1 & 9 & -9 \\ 0 & 0 & 1 & -1 & 3 & -2 \end{array}\right]$$

$$\xrightarrow{-R_2 \to R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 & -9 & 9 \\ 0 & 0 & 1 & -1 & 3 & -2 \end{array}\right]$$

Let $x_2 = s, x_5 = t$. Then from the second row, $x_4 - 9x_5 = 9 \implies x_4 = 9 + 9t$. From the third row,

$$x_3 - x_4 + 3x_5 = -2 \implies x_3 = (9 + 9t) - 3t - 2 = 7 + 6t.$$

From the first row,

$$x_1 - x_2 + x_3 + 2x_4 - x_5 = 2$$

so

$$x_1 = 2 + s - (7 + 6t) - 2(9 + 9t) + t = s - 23 - 23t.$$

Therefore,

$$\boxed{(x_1, x_2, x_3, x_4, x_5) = (s-23-23t,\ s,\ 7+6t,\ 9+9t,\ t), \qquad s, t \in \mathbb{R}.}$$

(h). $\begin{cases} 3x_1 - x_2 + x_3 - x_4 + 2x_5 = 5 \\ x_1 - x_2 - x_3 - 2x_4 - x_5 = 2 \\ 5x_1 - 2x_2 + x_3 - 3x_4 + 3x_5 = 10 \\ 2x_1 - x_2 - 2x_4 + x_5 = 5 \end{cases}$

*Xingjian's Solution*:

$$(A|b) = \left[\begin{array}{ccccc|c} 3 & -1 & 1 & -1 & 2 & 5 \\ 1 & -1 & -1 & -2 & -1 & 2 \\ 5 & -2 & 1 & -3 & 3 & 10 \\ 2 & -1 & 0 & -2 & 1 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -2 & -1 & 2 \\ 3 & -1 & 1 & -1 & 2 & 5 \\ 5 & -2 & 1 & -3 & 3 & 10 \\ 2 & -1 & 0 & -2 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{R_2-3R_1,\ R_3-5R_1,\ R_4-2R_1} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -2 & -1 & 2 \\ 0 & 2 & 4 & 5 & 5 & -1 \\ 0 & 3 & 6 & 7 & 8 & 0 \\ 0 & 1 & 2 & 2 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2-2R_4,\ R_3-3R_4} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -2 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 1 & 2 & 2 & 3 & 1 \end{array}\right] \xrightarrow{R_3-R_2,\ R_2\leftrightarrow R_4} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -2 & -1 & 2 \\ 0 & 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = s, \quad x_5 = t$. Then from the third row, $x_4 - x_5 = -3 \quad \Longrightarrow \quad x_4 = t - 3$. From the second row,

$$x_2 + 2x_3 + 2x_4 + 3x_5 = 1$$

so

$$x_2 = 1 - 2s - 2(t-3) - 3t = 7 - 2s - 5t.$$

From the first row,

$$x_1 - x_2 - x_3 - 2x_4 - x_5 = 2$$

thus

$$x_1 = 2 + x_2 + x_3 + 2x_4 + x_5 = 2 + (7 - 2s - 5t) + s + 2(t-3) + t = 3 - s - 2t.$$

Therefore,

$$\boxed{(x_1, x_2, x_3, x_4, x_5) = (3 - s - 2t,\ 7 - 2s - 5t,\ s,\ t - 3,\ t), \qquad s, t \in \mathbb{R}.}$$

**Question 5:** Let the reduced row echelon form of $A$ be

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$

Determine $A$ if the first, second, and fourth columns of $A$ are

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

respectively.

*Xingjian's Solution:*

Let the columns of $A$ be denoted by

$$A = \begin{pmatrix} | & | & | & | & | \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ | & | & | & | & | \end{pmatrix}.$$

Since the reduced row echelon form of $A$ is

$$R = \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix},$$

the pivot columns are columns $1, 2, 4$. Hence the non-pivot columns are linear combinations of the pivot columns with the same coefficients as in $R$. Therefore,

$$a_3 = 2a_1 - 5a_2, \qquad a_5 = -2a_1 - 3a_2 + 6a_4.$$

Now use the given columns:

$$a_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad a_4 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

First,

$$a_3 = 2a_1 - 5a_2 = 2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - 5\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Next,

$$a_5 = -2a_1 - 3a_2 + 6a_4 = -2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + 6\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

So

$$a_5 = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} + \begin{pmatrix} 6 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ -9 \end{pmatrix}.$$

Thus,

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{pmatrix}.$$

$$\boxed{A = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{pmatrix}}$$

**Question 7:** It can be shown that the vectors

$$u_1 = (2, -3, 1), \quad u_2 = (1, 4, -2), \quad u_3 = (-8, 12, -4), \quad u_4 = (1, 37, -17), \quad u_5 = (-3, -5, 8)$$

generate $\mathbb{R}^3$. Find a subset of $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

*Xingjian's Solution:*

We seek three linearly independent vectors from the given set.

First observe that $u_3 = (-8, 12, -4) = -4(2, -3, 1) = -4u_1$, so $u_3$ is linearly dependent on $u_1$. Hence $u_3$ cannot be part of a basis together with $u_1$.

Now test the set $\{u_1, u_2, u_5\}$. Form the matrix whose columns are these vectors:

$$M = \begin{pmatrix} 2 & 1 & -3 \\ -3 & 4 & -5 \\ 1 & -2 & 8 \end{pmatrix}.$$

Its determinant is

$$\det(M) = 2\begin{vmatrix} 4 & -5 \\ -2 & 8 \end{vmatrix} - 1\begin{vmatrix} -3 & -5 \\ 1 & 8 \end{vmatrix} + (-3)\begin{vmatrix} -3 & 4 \\ 1 & -2 \end{vmatrix}.$$

Compute each minor:

$$\begin{vmatrix} 4 & -5 \\ -2 & 8 \end{vmatrix} = 32 - 10 = 22,$$

$$\begin{vmatrix} -3 & -5 \\ 1 & 8 \end{vmatrix} = -24 + 5 = -19,$$

$$\begin{vmatrix} -3 & 4 \\ 1 & -2 \end{vmatrix} = 6 - 4 = 2.$$

Thus

$$\det(M) = 2(22) - 1(-19) + (-3)(2) = 44 + 19 - 6 = 57 \neq 0.$$

Therefore $u_1, u_2, u_5$ are linearly independent.

Since they are three linearly independent vectors in $\mathbb{R}^3$, they form a basis for $\mathbb{R}^3$.

Hence one possible basis is

$$\boxed{\{u_1, u_2, u_5\} = \{(2, -3, 1), (1, 4, -2), (-3, -5, 8)\}.}$$

**Question 9:** Let $W$ be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of the symmetric $2 \times 2$ matrices. The set

$$S = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \right\}$$

generates $W$. Find a subset of $S$ that is a basis for $W$.

*Xingjian's Solution:*

A general symmetric $2 \times 2$ matrix has the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

so $W$ has dimension 3. Therefore, to find a basis for $W$, it suffices to find 3 linearly independent matrices from $S$.

Let

$$A_1 = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}.$$

We check whether $A_1, A_2, A_3$ are linearly independent.

Suppose

$$c_1A_1 + c_2A_2 + c_3A_3 = 0.$$

Then

$$c_1\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} + c_2\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + c_3\begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Comparing entries gives

$$c_2 + 2c_3 = 0,$$
$$-c_1 + 2c_2 + c_3 = 0,$$
$$c_1 + 3c_2 + 9c_3 = 0.$$

From the first equation,

$$c_2 = -2c_3.$$

Substitute into the second equation:

$$-c_1 + 2(-2c_3) + c_3 = 0 \quad \Longrightarrow \quad -c_1 - 3c_3 = 0 \quad \Longrightarrow \quad c_1 = -3c_3.$$

Substitute into the third equation:

$$(-3c_3) + 3(-2c_3) + 9c_3 = 0 \quad \Longrightarrow \quad 0 = 0.$$

So this choice is dependent. So we should further try other cases.

Now try

$$A_1 = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

Suppose

$$c_1A_1 + c_2A_2 + c_4A_4 = 0.$$

Then

$$c_1\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} + c_2\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + c_4\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Comparing entries yields

$$c_2 + c_4 = 0,$$
$$-c_1 + 2c_2 - 2c_4 = 0,$$
$$c_1 + 3c_2 + 4c_4 = 0.$$

From the first equation,

$$c_4 = -c_2.$$

Substitute into the second equation:

$$-c_1 + 2c_2 - 2(-c_2) = 0 \quad \Longrightarrow \quad -c_1 + 4c_2 = 0 \quad \Longrightarrow \quad c_1 = 4c_2.$$

Substitute into the third equation:

$$4c_2 + 3c_2 + 4(-c_2) = 0 \quad \Longrightarrow \quad 3c_2 = 0,$$

so

$$c_2 = 0, \qquad c_4 = 0, \qquad c_1 = 0.$$

Hence $A_1, A_2, A_4$ are linearly independent.

Since $W$ has dimension 3, these three matrices form a basis of $W$.

Therefore, one possible basis is

$$\boxed{\left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \right\}.}$$

**Question 10:** Let

$$V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0\}.$$

(a). Show that $S = \{(0, 1, 1, 1, 0)\}$ is a linearly independent subset of $V$.

(b). Extend $S$ to a basis for $V$.

*Xingjian's Solution:*

**(a)** Let $v_1 = (0, 1, 1, 1, 0)$. First we verify that $v_1 \in V$. Substitute into the defining equation of $V$: $0 - 2(1) + 3(1) - 1 + 2(0) = 0$. Thus $v_1 \in V$.

Since $S$ contains only one nonzero vector, it is automatically linearly independent. Therefore $S = \{(0, 1, 1, 1, 0)\}$ is a linearly independent subset of $V$.

**(b)** We now extend $S$ to a basis for $V$.

The condition defining $V$ is $x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0$, so

$$x_1 = 2x_2 - 3x_3 + x_4 - 2x_5.$$

Hence every vector in $V$ has the form

$$(x_1, x_2, x_3, x_4, x_5) = (2x_2 - 3x_3 + x_4 - 2x_5,\ x_2,\ x_3,\ x_4,\ x_5).$$

Writing this in parametric form,

$$(x_1, x_2, x_3, x_4, x_5) = x_2(2, 1, 0, 0, 0) + x_3(-3, 0, 1, 0, 0) + x_4(1, 0, 0, 1, 0) + x_5(-2, 0, 0, 0, 1).$$

Thus $V$ is spanned by $(2, 1, 0, 0, 0)$, $(-3, 0, 1, 0, 0)$, $(1, 0, 0, 1, 0)$, $(-2, 0, 0, 0, 1)$. Since $\dim(V) = 4$, we need 4 linearly independent vectors in total.

We already have $v_1 = (0, 1, 1, 1, 0)$. Now choose three additional vectors in $V$, for example

$$v_2 = (2, 1, 0, 0, 0), \qquad v_3 = (-3, 0, 1, 0, 0), \qquad v_4 = (-2, 0, 0, 0, 1).$$

We check linear independence. Suppose $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$. That is,

$$c_1(0, 1, 1, 1, 0) + c_2(2, 1, 0, 0, 0) + c_3(-3, 0, 1, 0, 0) + c_4(-2, 0, 0, 0, 1) = 0.$$

Comparing coordinates gives

$$2c_2 - 3c_3 - 2c_4 = 0,$$

$c_1 + c_2 = 0$, $c_1 + c_3 = 0$, $c_1 = 0$, $c_4 = 0$. From $c_1 = 0$, we get $c_2 = 0$ and $c_3 = 0$, and also $c_4 = 0$. Hence the four vectors are linearly independent.

Therefore, a basis for $V$ extending $S$ is

$$\boxed{\{(0,1,1,1,0),\ (2,1,0,0,0),\ (-3,0,1,0,0),\ (-2,0,0,0,1)\}\,.}$$

**Question 11:** Let $V$ be as in Exercise 10.
(a). Show that $S = \{(1,2,1,0,0)\}$ is a linearly independent subset of $V$.
(b). Extend $S$ to a basis for $V$.
*Xingjian's Solution*:
**(a)** Let $v_1 = (1,2,1,0,0)$. We first verify that $v_1 \in V$. Substitute into the equation

$$x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0 :$$

$1 - 2(2) + 3(1) - 0 + 2(0) = 1 - 4 + 3 = 0$. Thus $v_1 \in V$.

Since $S$ contains only one nonzero vector, it is linearly independent. Therefore $S = \{(1,2,1,0,0)\}$ is a linearly independent subset of $V$.

**(b)** As in Question 10, every vector in $V$ has the form

$$(x_1,x_2,x_3,x_4,x_5) = x_2(2,1,0,0,0) + x_3(-3,0,1,0,0) + x_4(1,0,0,1,0) + x_5(-2,0,0,0,1).$$

Hence $\dim(V) = 4$.

We already have $v_1 = (1,2,1,0,0)$. Now choose three more vectors in $V$, for example

$$v_2 = (2,1,0,0,0), \qquad v_3 = (1,0,0,1,0), \qquad v_4 = (-2,0,0,0,1).$$

Each of these lies in $V$.

Now suppose $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$. Then

$$c_1(1,2,1,0,0) + c_2(2,1,0,0,0) + c_3(1,0,0,1,0) + c_4(-2,0,0,0,1) = 0.$$

Comparing coordinates yields

$$c_1 + 2c_2 + c_3 - 2c_4 = 0,$$

$2c_1 + c_2 = 0$, $c_1 = 0$, $c_3 = 0$, $c_4 = 0$. From $c_1 = 0$, the second equation gives $c_2 = 0$. Hence $c_1 = c_2 = c_3 = c_4 = 0$, so the four vectors are linearly independent.

Therefore, a basis for $V$ extending $S$ is

$$\boxed{\{(1,2,1,0,0),\ (2,1,0,0,0),\ (1,0,0,1,0),\ (-2,0,0,0,1)\}\,.}$$

**Question 12:** Let $V$ denote the set of all solutions to the system of linear equations

$$x_1 - x_2 + 2x_4 - 3x_5 + x_6 = 0,$$
$$2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 0.$$

(a). Show that $S = \{(0,-1,0,1,1,0),(1,0,1,1,1,0)\}$ is a linearly independent subset of $V$.

(b). Extend $S$ to a basis for $V$.

*Xingjian's Solution:*

**(a)** Let $v_1 = (0,-1,0,1,1,0), v_2 = (1,0,1,1,1,0)$. We first verify that both vectors belong to $V$.

For $v_1 = (0,-1,0,1,1,0)$, substitute into the equations:

$$x_1 - x_2 + 2x_4 - 3x_5 + x_6 = 0 - (-1) + 2(1) - 3(1) + 0 = 1 + 2 - 3 = 0,$$

and

$$2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 2(0) - (-1) - 0 + 3(1) - 4(1) + 4(0) = 1 + 3 - 4 = 0.$$

Hence $v_1 \in V$.

For $v_2 = (1,0,1,1,1,0)$, we compute

$$x_1 - x_2 + 2x_4 - 3x_5 + x_6 = 1 - 0 + 2(1) - 3(1) + 0 = 1 + 2 - 3 = 0,$$

and

$$2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 2(1) - 0 - 1 + 3(1) - 4(1) + 4(0) = 2 - 1 + 3 - 4 = 0.$$

Hence $v_2 \in V$.

Now suppose $c_1v_1 + c_2v_2 = 0$. Then $c_1(0,-1,0,1,1,0) + c_2(1,0,1,1,1,0) = 0$. Comparing the first coordinate gives $c_2 = 0$. Comparing the second coordinate then gives $-c_1 = 0$, so $c_1 = 0$. Therefore $v_1$ and $v_2$ are linearly independent.

Thus $S = \{(0,-1,0,1,1,0),\ (1,0,1,1,1,0)\}$ is a linearly independent subset of $V$.

**(b)** We now extend $S$ to a basis for $V$.

The system is

$$\begin{cases} x_1 - x_2 + 2x_4 - 3x_5 + x_6 = 0, \\ 2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 0. \end{cases}$$

We solve for $x_1$ and $x_3$ in terms of the free variables $x_2, x_4, x_5, x_6$.

From the first equation,

$$x_1 = x_2 - 2x_4 + 3x_5 - x_6.$$

Substitute this into the second equation:

$$2(x_2 - 2x_4 + 3x_5 - x_6) - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 0.$$

Simplifying,

$$x_2 - x_4 + 2x_5 + 2x_6 - x_3 = 0,$$

so

$$x_3 = x_2 - x_4 + 2x_5 + 2x_6.$$

Hence every vector in $V$ has the form

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (x_2 - 2x_4 + 3x_5 - x_6,\ x_2,\ x_2 - x_4 + 2x_5 + 2x_6,\ x_4,\ x_5,\ x_6).$$

Thus

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_6) = x_2(1,1,1,0,0,0) + x_4(-2,0,-1,1,0,0) \\ + x_5(3,0,2,0,1,0) + x_6(-1,0,2,0,0,1).\end{aligned}$$

So $\dim(V) = 4$.

We already have two linearly independent vectors

$$v_1 = (0,-1,0,1,1,0), \qquad v_2 = (1,0,1,1,1,0).$$

Choose two additional vectors in $V$, for example

$$v_3 = (1,1,1,0,0,0), \qquad v_4 = (-1,0,2,0,0,1).$$

Now suppose

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0.$$

That is,

$$c_1(0,-1,0,1,1,0) + c_2(1,0,1,1,1,0) + c_3(1,1,1,0,0,0) + c_4(-1,0,2,0,0,1) = 0.$$

Comparing coordinates gives

$$\begin{aligned} c_2 + c_3 - c_4 &= 0, \\ -c_1 + c_3 &= 0, \\ c_2 + c_3 + 2c_4 &= 0, \\ c_1 + c_2 &= 0, \\ c_1 + c_2 &= 0, \\ c_4 &= 0. \end{aligned}$$

From $c_4 = 0$, the first and third equations become $c_2 + c_3 = 0, c_2 + c_3 = 0$. Also, from $-c_1 + c_3 = 0$, we get $c_3 = c_1$, and from $c_1 + c_2 = 0$, we get $c_2 = -c_1$. Hence

$$c_2 + c_3 = -c_1 + c_1 = 0.$$

Thus all relations reduce consistently, and since $c_4 = 0$, $c_2 = -c_1$, $c_3 = c_1$, and using $c_1 + c_2 = 0$, we conclude the only solution is $c_1 = c_2 = c_3 = c_4 = 0$. Therefore $v_1, v_2, v_3, v_4$ are linearly independent.

Hence a basis for $V$ extending $S$ is

$$\boxed{\{(0,-1,0,1,1,0),\ (1,0,1,1,1,0),\ (1,1,1,0,0,0),\ (-1,0,2,0,0,1)\}.}$$

**Question 13:** Let $V$ be as in Exercise 12.

(a).Show that $S = \{(1,0,1,1,1,0), (0,2,1,1,0,0)\}$ is a linearly independent subset of $V$.

(b). Extend $S$ to a basis for $V$.

*Xingjian's Solution:*

**(a)** Let $v_1 = (1,0,1,1,1,0), v_2 = (0,2,1,1,0,0)$. We verify that both vectors belong to $V$.

For $v_1 = (1,0,1,1,1,0)$, $x_1 - x_2 + 2x_4 - 3x_5 + x_6 = 1 - 0 + 2(1) - 3(1) + 0 = 1 + 2 - 3 = 0$, and

$$2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 2(1) - 0 - 1 + 3(1) - 4(1) + 0 = 2 - 1 + 3 - 4 = 0.$$

So $v_1 \in V$.

For $v_2 = (0,2,1,1,0,0)$,

$$x_1 - x_2 + 2x_4 - 3x_5 + x_6 = 0 - 2 + 2(1) - 0 + 0 = 0,$$

and

$$2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 0 - 2 - 1 + 3(1) - 0 + 0 = 0.$$

So $v_2 \in V$.

Now suppose $c_1v_1 + c_2v_2 = 0$. Then $c_1(1,0,1,1,1,0) + c_2(0,2,1,1,0,0) = 0$. Comparing the first coordinate gives $c_1 = 0$. Comparing the second coordinate then gives $2c_2 = 0$, so $c_2 = 0$. Therefore $v_1$ and $v_2$ are linearly independent.

Thus $S = \{(1,0,1,1,1,0),\ (0,2,1,1,0,0)\}$ is a linearly independent subset of $V$.

**(b)** As in Question 12, every vector in $V$ has the form

$$(x_1,x_2,x_3,x_4,x_5,x_6) = x_2(1,1,1,0,0,0) + x_4(-2,0,-1,1,0,0) + x_5(3,0,2,0,1,0) + x_6(-1,0,2,0,0,1).$$

Hence $\dim(V) = 4$.

We already have $v_1 = (1,0,1,1,1,0), v_2 = (0,2,1,1,0,0)$. Choose two more vectors in $V$, for example $v_3 = (1,1,1,0,0,0), v_4 = (-1,0,2,0,0,1)$. Now suppose $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$. That is,

$$c_1(1,0,1,1,1,0) + c_2(0,2,1,1,0,0) + c_3(1,1,1,0,0,0) + c_4(-1,0,2,0,0,1) = 0.$$

Comparing coordinates yields

$$\begin{aligned} c_1 + c_3 - c_4 &= 0, \\ 2c_2 + c_3 &= 0, \\ c_1 + c_2 + c_3 + 2c_4 &= 0, \\ c_1 + c_2 &= 0, \\ c_1 &= 0, \\ c_4 &= 0. \end{aligned}$$

From $c_1 = 0$ and $c_4 = 0$, the first equation gives $c_3 = 0$. Then the second equation gives $2c_2 = 0$, so $c_2 = 0$. Therefore $c_1 = c_2 = c_3 = c_4 = 0$, and thus $v_1, v_2, v_3, v_4$ are linearly independent.

Hence a basis for $V$ extending $S$ is

$$\boxed{\{(1,0,1,1,1,0),\ (0,2,1,1,0,0),\ (1,1,1,0,0,0),\ (-1,0,2,0,0,1)\}.}$$

MAS 4105 Solutions for Section 3.4

*Xingjian Xu*
xingjianxu@ufl.edu
LIT 453
Webpage: https://people.clas.ufl.edu/xingjianxu/mas4105-26springs/
Office Hours: Tuesday, Periods 3 and 5 on LIT453;