Question 4: For each system with invertible coefficient matrix A,
(1) Compute A^{-1},
(2) Use A^{-1} to solve the system.

(b)

$$x_1 + 2x_2 - x_3 = 5, \quad x_1 + x_2 + x_3 = 1, \quad 2x_1 - 2x_2 + x_3 = 4$$

Xingjian's Solution:
We first write the system in matrix form:

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}.$$

We will compute A^{-1} by row reducing the augmented matrix $[A \mid I]$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right].$$

Step 1: Eliminate the entries below the first pivot.
Use

$$R_2 \leftarrow R_2 - R_1, \qquad R_3 \leftarrow R_3 - 2R_1.$$

Then

$$R_2 - R_1 = (1, 1, 1 \mid 0, 1, 0) - (1, 2, -1 \mid 1, 0, 0) = (0, -1, 2 \mid -1, 1, 0),$$

and

$$R_3 - 2R_1 = (2, -2, 1 \mid 0, 0, 1) - 2(1, 2, -1 \mid 1, 0, 0) = (0, -6, 3 \mid -2, 0, 1).$$

So the matrix becomes

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & -6 & 3 & -2 & 0 & 1 \end{array}\right].$$

Step 2: Make the second pivot equal to 1.
Use

$$R_2 \leftarrow -R_2.$$

Then

$$-R_2 = -(0, -1, 2 \mid -1, 1, 0) = (0, 1, -2 \mid 1, -1, 0).$$

Thus

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & -6 & 3 & -2 & 0 & 1 \end{array}\right].$$

Step 3: Eliminate the other entries in column 2.
Use

$$R_1 \leftarrow R_1 - 2R_2, \qquad R_3 \leftarrow R_3 + 6R_2.$$

First,

$$R_1 - 2R_2 = (1, 2, -1 \mid 1, 0, 0) - 2(0, 1, -2 \mid 1, -1, 0).$$

Compute:

$$2(0, 1, -2 \mid 1, -1, 0) = (0, 2, -4 \mid 2, -2, 0),$$

so

$$R_1 - 2R_2 = (1, 0, 3 \mid -1, 2, 0).$$

Next,

$$R_3 + 6R_2 = (0, -6, 3 \mid -2, 0, 1) + 6(0, 1, -2 \mid 1, -1, 0).$$

Compute:

$$6(0, 1, -2 \mid 1, -1, 0) = (0, 6, -12 \mid 6, -6, 0),$$

so

$$R_3 + 6R_2 = (0, 0, -9 \mid 4, -6, 1).$$

Hence the matrix is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 2 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & -9 & 4 & -6 & 1 \end{array}\right].$$

Step 4: Make the third pivot equal to 1.
Use

$$R_3 \leftarrow -\frac{1}{9}R_3.$$

Then

$$-\frac{1}{9}(0, 0, -9 \mid 4, -6, 1) = \left(0, 0, 1 \mid -\frac{4}{9}, \frac{2}{3}, -\frac{1}{9}\right).$$

So we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 2 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{array}\right].$$

Step 5: Eliminate the entries above the third pivot.
Use

$$R_1 \leftarrow R_1 - 3R_3, \qquad R_2 \leftarrow R_2 + 2R_3.$$

First,

$$R_1 - 3R_3 = (1, 0, 3 \mid -1, 2, 0) - 3\left(0, 0, 1 \mid -\frac{4}{9}, \frac{2}{3}, -\frac{1}{9}\right).$$

Compute

$$3R_3 = \left(0, 0, 3 \mid -\frac{4}{3}, 2, -\frac{1}{3}\right),$$

so

$$R_1 - 3R_3 = \left(1,0,0 \mid -1+\frac{4}{3},\ 2-2,\ 0+\frac{1}{3}\right) = \left(1,0,0 \mid \frac{1}{3},0,\frac{1}{3}\right).$$

Next,

$$R_2 + 2R_3 = (0,1,-2 \mid 1,-1,0) + 2\left(0,0,1 \mid -\frac{4}{9},\frac{2}{3},-\frac{1}{9}\right).$$

Compute

$$2R_3 = \left(0,0,2 \mid -\frac{8}{9},\frac{4}{3},-\frac{2}{9}\right),$$

so

$$R_2 + 2R_3 = \left(0,1,0 \mid 1-\frac{8}{9},\ -1+\frac{4}{3},\ 0-\frac{2}{9}\right) = \left(0,1,0 \mid \frac{1}{9},\frac{1}{3},-\frac{2}{9}\right).$$

Therefore,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ 0 & 0 & 1 & -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{array}\right].$$

Since the left side is now the identity matrix, the right side is A^{-1}. Hence

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{bmatrix}.$$

Step 6: Compute $\mathbf{x} = A^{-1}\mathbf{b}$.
Now multiply:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}.$$

Compute each entry carefully.
For x_1,

$$x_1 = \frac{1}{3}(5) + 0(1) + \frac{1}{3}(4) = \frac{5}{3} + \frac{4}{3} = \frac{9}{3} = 3.$$

For x_2,

$$x_2 = \frac{1}{9}(5) + \frac{1}{3}(1) - \frac{2}{9}(4) = \frac{5}{9} + \frac{3}{9} - \frac{8}{9} = 0.$$

For x_3,

$$x_3 = -\frac{4}{9}(5) + \frac{2}{3}(1) - \frac{1}{9}(4) = -\frac{20}{9} + \frac{6}{9} - \frac{4}{9} = -\frac{18}{9} = -2.$$

Thus

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}.$$

So the solution to the system is

$$\boxed{x_1 = 3, \qquad x_2 = 0, \qquad x_3 = -2.}$$

Check:
Substitute into the original equations:

$$x_1 + 2x_2 - x_3 = 3 + 2(0) - (-2) = 5,$$
$$x_1 + x_2 + x_3 = 3 + 0 + (-2) = 1,$$
$$2x_1 - 2x_2 + x_3 = 2(3) - 2(0) + (-2) = 4.$$

All three equations are satisfied, so the answer is correct.

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