Question 1: Label the following statements as true or false.

(a). Any system of linear equations has at least one solution.

(b). Any system of linear equations has at most one solution.

(c). A homogeneous system of linear equations has at least one solution.

(d). Any system of n linear equations in n unknowns has at most one solution.

(e). Any system of n linear equations in n unknowns has at least one solution.

(f). If the homogeneous system corresponding to a given system has a solution, then the given system has a solution.

(g). If the coefficient matrix of a homogeneous system in n unknowns is invertible, then the system has no nonzero solutions.

(h). The solution set of any system of m linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Xingjian's Solution:

(a). **False**. A system can be inconsistent.

(b). **False**. A system may have infinitely many solutions.

(c). **True**. The trivial solution always exists.

(d). **False**. The system may have infinitely many or no solutions.

(e). **False**. The system may be inconsistent.

(f). **True**. The homogeneous system always has the trivial solution, but this does not guarantee the original system has a solution unless consistent; hence statement is False.

(g). **True**. If invertible, only the trivial solution exists.

(h). **False**. Only homogeneous systems have solution sets that are subspaces.

Question 2: For each of the following homogeneous systems, find the dimension and a basis for the solution set.

(a) $x_1 + 3x_2 = 0, \qquad 2x_1 + 6x_2 = 0$

Xingjian's Solution:

The second equation is just twice the first, so there is only one independent equation: $x_1 = -3x_2$. Let $x_2 = t$. Then $(x_1, x_2) = (-3t, t) = t(-3, 1)$. So a basis for the solution set is $\{(-3, 1)\}$, and the dimension is $\boxed{1}$.

(b) $x_1 + x_2 - x_3 = 0, \qquad 4x_1 + x_2 - 2x_3 = 0$

Xingjian's Solution:

From the first equation, $x_1 = -x_2 + x_3$.

Substitute into the second: $4(-x_2 + x_3) + x_2 - 2x_3 = 0$ $x_2 = \frac{2}{3}x_3$.

Then $x_1 = -\frac{2}{3}x_3 + x_3 = \frac{1}{3}x_3$. Let $x_3 = 3t$. Then $(x_1, x_2, x_3) = t(1, 2, 3)$. So a basis is $\{(1, 2, 3)\}$, and the dimension is $\boxed{1}$.

(c) $x_1 + 2x_2 - x_3 = 0, \qquad 2x_1 + x_2 + x_3 = 0$

Xingjian's Solution:

From the first equation, $x_1 = -2x_2 + x_3$. Substitute into the second: $2(-2x_2 + x_3) + x_2 + x_3 = 0$ That means $x_2 = x_3$. Then $x_1 = -2x_3 + x_3 = -x_3$. Let $x_3 = t$. Then $(x_1, x_2, x_3) = t(-1, 1, 1)$. So a basis is $\{(-1, 1, 1)\}$, and the dimension is $\boxed{1}$.

(d) $2x_1 + x_2 - x_3 = 0, \qquad x_1 - x_2 + x_3 = 0, \qquad x_1 + 2x_2 - 2x_3 = 0$

Xingjian's Solution:
From $x_1 - x_2 + x_3 = 0$, we get $x_1 = x_2 - x_3$.
Substitute into the first equation: $2(x_2 - x_3) + x_2 - x_3 = 0$ We get $x_2 = x_3$.
Hence $x_1 = x_2 - x_3 = 0$.
So every solution has the form $(x_1, x_2, x_3) = (0, t, t) = t(0, 1, 1)$. Check in the third equation: $0 + 2t - 2t = 0$, so it works. Thus a basis is $\{(0, 1, 1)\}$, and the dimension is $\boxed{1}$.

(e)

$$x_1 + 2x_2 - 3x_3 + x_4 = 0$$

Xingjian's Solution:
There is one equation in four variables, so there are three free variables.
Solve for x_1: $x_1 = -2x_2 + 3x_3 - x_4$. Let $x_2 = s, x_3 = t, x_4 = u$. Then $(x_1, x_2, x_3, x_4) = (-2s + 3t - u, s, t, u)$. Write this as

$$(x_1, x_2, x_3, x_4) = s(-2, 1, 0, 0) + t(3, 0, 1, 0) + u(-1, 0, 0, 1).$$

So a basis is $\{(-2, 1, 0, 0), (3, 0, 1, 0), (-1, 0, 0, 1)\}$, and the dimension is $\boxed{3}$.

(f) $x_1 + 2x_2 = 0, \qquad x_1 - x_2 = 0$
Xingjian's Solution:
From the second equation, $x_1 = x_2$.
Substitute into the first: $x_2 + 2x_2 = 0$ That means $x_2 = 0$. Hence $x_1 = 0$.
So the only solution is the trivial solution: $(0, 0)$. Therefore the solution set has basis $\emptyset$ and dimension $\boxed{0}$.

(g) $x_1 + 2x_2 + x_3 + x_4 = 0, \qquad x_2 - x_3 + x_4 = 0$
Xingjian's Solution:
From the second equation, $x_2 = x_3 - x_4$. Substitute into the first: $x_1 + 2(x_3 - x_4) + x_3 + x_4 = 0$ That will give you $x_1 = -3x_3 + x_4$. Let $x_3 = s, x_4 = t$. Then

$$(x_1, x_2, x_3, x_4) = (-3s + t, s - t, s, t).$$

Write this as $(x_1, x_2, x_3, x_4) = s(-3, 1, 1, 0) + t(1, -1, 0, 1)$. So a basis is $\{(-3, 1, 1, 0), (1, -1, 0, 1)\}$, and the dimension is $\boxed{2}$.

Question 3: Using the results of Exercise 2, find all solutions to the following systems.

(a) Solve

$$\begin{cases} x_1 + 3x_2 = 5, \\ 2x_1 + 6x_2 = 10. \end{cases}$$

Xingjian's Solution:
First consider the corresponding homogeneous system

$$\begin{cases} x_1 + 3x_2 = 0, \\ 2x_1 + 6x_2 = 0. \end{cases}$$

Since the second equation is twice the first, this reduces to $x_1 + 3x_2 = 0$. Let $x_2 = t$. Then $x_1 = -3t$. So the homogeneous solution is

$$(x_1, x_2) = t(-3, 1), \qquad t \in \mathbb{R}.$$

Now find a particular solution of the nonhomogeneous system. Again, the second equation is twice the first, so it is enough to solve $x_1 + 3x_2 = 5$. Choose $x_2 = 0$. Then $x_1 = 5$. Thus one particular solution is

$$(x_1, x_2) = (5, 0).$$

Therefore all solutions are

$$(x_1, x_2) = (5, 0) + t(-3, 1), \qquad t \in \mathbb{R}.$$

(b) Solve

$$\begin{cases} x_1 + x_2 - x_3 = 1, \\ 4x_1 + x_2 - 2x_3 = 3. \end{cases}$$

Xingjian's Solution:

First consider the corresponding homogeneous system

$$\begin{cases} x_1 + x_2 - x_3 = 0, \\ 4x_1 + x_2 - 2x_3 = 0. \end{cases}$$

Subtract the first equation from the second:

$$(4x_1 + x_2 - 2x_3) - (x_1 + x_2 - x_3) = 0,$$

so $3x_1 - x_3 = 0$, which gives $x_3 = 3x_1$.

Substitute into the first equation: $x_1 + x_2 - 3x_1 = 0$, hence $x_2 = 2x_1$.

Let $x_1 = t$. Then $(x_1, x_2, x_3) = t(1, 2, 3)$. So the homogeneous solution is $(x_1, x_2, x_3) = t(1, 2, 3), t \in \mathbb{R}$.

Now find a particular solution of the nonhomogeneous system. Choose $x_3 = 0$. Then we solve

$$x_1 + x_2 = 1, \qquad 4x_1 + x_2 = 3.$$

Subtract the first equation from the second: $3x_1 = 2$, so $x_1 = \frac{2}{3}$. Then $x_2 = 1 - \frac{2}{3} = \frac{1}{3}$. Thus one particular solution is $\left(\frac{2}{3}, \frac{1}{3}, 0\right)$.

Therefore all solutions are

$$(x_1, x_2, x_3) = \left(\frac{2}{3}, \frac{1}{3}, 0\right) + t(1, 2, 3), \qquad t \in \mathbb{R}.$$

(c) Solve

$$\begin{cases} x_1 + 2x_2 - x_3 = 3, \\ 2x_1 + x_2 + x_3 = 6. \end{cases}$$

Xingjian's Solution:

First consider the corresponding homogeneous system

$$\begin{cases} x_1 + 2x_2 - x_3 = 0, \\ 2x_1 + x_2 + x_3 = 0. \end{cases}$$

Add the two equations: $3x_1 + 3x_2 = 0$, so $x_1 = -x_2$.

Substitute into the first equation: $-x_2 + 2x_2 - x_3 = 0$, hence $x_3 = x_2$.

Let $x_2 = t$. Then $x_1 = -t, x_3 = t$. So the homogeneous solution is $(x_1, x_2, x_3) = t(-1, 1, 1), t \in \mathbb{R}$.

Now find a particular solution of the nonhomogeneous system. Choose $x_3 = 0$. Then

$$x_1 + 2x_2 = 3, \qquad 2x_1 + x_2 = 6.$$

Multiply the first equation by 2:

$$2x_1 + 4x_2 = 6.$$

Subtract the second equation:

$$3x_2 = 0,$$

so $x_2 = 0$. Then $x_1 = 3$. Thus one particular solution is $(3, 0, 0)$.

Therefore all solutions are $(x_1, x_2, x_3) = (3, 0, 0) + t(-1, 1, 1), \qquad t \in \mathbb{R}$.

(d) Solve

$$\begin{cases} 2x_1 + x_2 - x_3 = 5, \\ x_1 - x_2 + x_3 = 1, \\ x_1 + 2x_2 - 2x_3 = 4. \end{cases}$$

Xingjian's Solution:

First consider the corresponding homogeneous system

$$\begin{cases} 2x_1 + x_2 - x_3 = 0, \\ x_1 - x_2 + x_3 = 0, \\ x_1 + 2x_2 - 2x_3 = 0. \end{cases}$$

Add the first two equations: $3x_1 = 0$, so $x_1 = 0$. Then the second equation becomes $-x_2 + x_3 = 0$, hence $x_3 = x_2$.

Let $x_2 = t$. Then $x_1 = 0, x_3 = t$. So the homogeneous solution is $(x_1, x_2, x_3) = t(0, 1, 1), \quad t \in \mathbb{R}$.

Now find a particular solution of the nonhomogeneous system

$$\begin{cases} 2x_1 + x_2 - x_3 = 5, \\ x_1 - x_2 + x_3 = 1, \\ x_1 + 2x_2 - 2x_3 = 4. \end{cases}$$

Add the first two equations: $3x_1 = 6$, so $x_1 = 2$. Substitute into the second equation:

$$2 - x_2 + x_3 = 1,$$

hence $x_3 = x_2 - 1$.

Substitute $x_1 = 2$ into the third equation: $2 + 2x_2 - 2x_3 = 4$, so $x_2 - x_3 = 1$, which is the same relation. Choose $x_2 = 1$. Then $x_3 = 0$.

Thus one particular solution is $(2, 1, 0)$.

Therefore all solutions are $(x_1, x_2, x_3) = (2, 1, 0) + t(0, 1, 1), \qquad t \in \mathbb{R}$.

(e) Solve

$$x_1 + 2x_2 - 3x_3 + x_4 = 1.$$

Xingjian's Solution:

First consider the corresponding homogeneous system

$$x_1 + 2x_2 - 3x_3 + x_4 = 0.$$

Solve for x_1: $x_1 = -2x_2 + 3x_3 - x_4$.

Let $x_2 = s, x_3 = t, x_4 = r$. Then $x_1 = -2s + 3t - r$.

Hence

$$(x_1, x_2, x_3, x_4) = s(-2, 1, 0, 0) + t(3, 0, 1, 0) + r(-1, 0, 0, 1).$$

So the homogeneous solution set is spanned by

$$(-2, 1, 0, 0), \qquad (3, 0, 1, 0), \qquad (-1, 0, 0, 1).$$

Now find a particular solution of the nonhomogeneous equation

$$x_1 + 2x_2 - 3x_3 + x_4 = 1.$$

Choose $x_2 = x_3 = x_4 = 0$. Then $x_1 = 1$. Thus one particular solution is $(1, 0, 0, 0)$.

Therefore all solutions are

$$(x_1, x_2, x_3, x_4) = (1, 0, 0, 0) + s(-2, 1, 0, 0) + t(3, 0, 1, 0) + r(-1, 0, 0, 1), \qquad r, s, t \in \mathbb{R}.$$

(f) Solve

$$\begin{cases} x_1 + 2x_2 = 5, \\ x_1 - x_2 = -1. \end{cases}$$

Xingjian's Solution:

First consider the corresponding homogeneous system

$$\begin{cases} x_1 + 2x_2 = 0, \\ x_1 - x_2 = 0. \end{cases}$$

Subtract the second equation from the first: $3x_2 = 0$, so $x_2 = 0$. Then $x_1 = 0$. Thus the homogeneous system has only the trivial solution: $(x_1, x_2) = (0, 0)$.

Now solve the nonhomogeneous system

$$\begin{cases} x_1 + 2x_2 = 5, \\ x_1 - x_2 = -1. \end{cases}$$

Subtract the second equation from the first: $3x_2 = 6$, so $x_2 = 2$.

Then from $x_1 - x_2 = -1$, $x_1 - 2 = -1$, hence $x_1 = 1$.

Thus a particular solution is $(1, 2)$.

Since the homogeneous solution is only the zero vector, the unique solution is

$$(x_1, x_2) = (1, 2).$$

(g) Solve

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 1, \\ x_2 - x_3 + x_4 = 1. \end{cases}$$

Xingjian's Solution:

First consider the corresponding homogeneous system

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0, \\ x_2 - x_3 + x_4 = 0. \end{cases}$$

From the second equation, $x_2 = x_3 - x_4$.

Substitute into the first equation: $x_1 + 2(x_3 - x_4) + x_3 + x_4 = 0$, so

$$x_1 + 3x_3 - x_4 = 0.$$

Hence $x_1 = -3x_3 + x_4$.

Let $x_3 = s, x_4 = t$. Then $x_2 = s - t, x_1 = -3s + t$. So

$$(x_1, x_2, x_3, x_4) = s(-3, 1, 1, 0) + t(1, -1, 0, 1).$$

Thus the homogeneous solution is $(x_1, x_2, x_3, x_4) = s(-3, 1, 1, 0) + t(1, -1, 0, 1), \quad s, t \in \mathbb{R}$.

Now find a particular solution of the nonhomogeneous system

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 1, \\ x_2 - x_3 + x_4 = 1. \end{cases}$$

Choose $x_3 = 0$ and $x_4 = 0$. Then the second equation gives $x_2 = 1$. Substitute into the first equation: $x_1 + 2 = 1$, so $x_1 = -1$. Thus one particular solution is $(-1, 1, 0, 0)$.

Therefore all solutions are

$$(x_1, x_2, x_3, x_4) = (-1, 1, 0, 0) + s(-3, 1, 1, 0) + t(1, -1, 0, 1), \qquad s, t \in \mathbb{R}.$$

Question 4: For each system with invertible coefficient matrix A,

(1) Compute A^{-1},
(2) Use A^{-1} to solve the system.

(a)

$$x_1 + 3x_2 = 4, \quad 2x_1 + 5x_2 = 3$$

Xingjian's Solution:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

$$\det(A) = 5 - 6 = -1.$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

Then

$$x = A^{-1}b = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -20 + 9 \\ 8 - 3 \end{bmatrix} = \begin{bmatrix} -11 \\ 5 \end{bmatrix}.$$

(b)

$$x_1 + 2x_2 - x_3 = 5, \quad x_1 + x_2 + x_3 = 1, \quad 2x_1 - 2x_2 + x_3 = 4$$

Xingjian's Solution:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}.$$

Compute A^{-1} (by row reduction or adjugate), obtaining

$$A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Then

$$x = A^{-1}b = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 - 2 + 4 \\ -5 + 3 - 4 \\ 5 - 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \\ 4 \end{bmatrix}.$$

Question 8: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(a, b, c) = (a + b,\ b - 2c,\ a + 2c).$$

For each vector $v \in \mathbb{R}^3$, determine whether $v \in R(T)$.

(a) $v = (1, 3, -2)$

Xingjian's Solution:

To determine whether $v \in R(T)$, we must decide whether there exist $a, b, c \in \mathbb{R}$ such that

$$T(a, b, c) = (1, 3, -2).$$

This gives the system

$$\begin{cases} a + b = 1, \\ b - 2c = 3, \\ a + 2c = -2. \end{cases}$$

From the first equation,

$$a = 1 - b.$$

Substitute this into the third equation:

$$1 - b + 2c = -2,$$

so

$$-b + 2c = -3.$$

Equivalently,

$$b - 2c = 3.$$

But this is exactly the second equation, so the system is consistent.

Now choose $c = 0$. Then from $b - 2c = 3$, we get

$$b = 3.$$

Then from $a + b = 1$, we obtain

$$a = 1 - 3 = -2.$$

Thus

$$T(-2, 3, 0) = (1, 3, -2).$$

Therefore, $(1, 3, -2) \in R(T)$.

(b) $v = (2, 1, 1)$

Xingjian's Solution: We must determine whether there exist $a, b, c \in \mathbb{R}$ such that

$$T(a, b, c) = (2, 1, 1).$$

This gives the system

$$\begin{cases} a + b = 2, \\ b - 2c = 1, \\ a + 2c = 1. \end{cases}$$

From the first equation, $a = 2 - b$.

Substitute this into the third equation:

$$2 - b + 2c = 1,$$

so

$$-b + 2c = -1.$$

Equivalently,

$$b - 2c = 1.$$

This is exactly the second equation, so the system is consistent.

Choose $c = 0$. Then from $b - 2c = 1$, we get

$$b = 1.$$

Then from $a + b = 2$, we obtain

$$a = 1.$$

Thus

$$T(1, 1, 0) = (2, 1, 1).$$

Therefore,

$$(2, 1, 1) \in R(T).$$

Question 9: Prove that the system of linear equations $Ax = b$ has a solution if and only if $b \in R(A)$.

Xingjian's Solution:

Recall that $R(A)$, the range of A, is the set of all vectors of the form Ax for some vector x.

First suppose that the system $Ax = b$ has a solution. Then there exists some vector x such that $Ax = b$. By the definition of the range of A, every vector of the form Ax belongs to $R(A)$. Hence $b \in R(A)$.

Conversely, suppose that $b \in R(A)$. Then, by definition of $R(A)$, there exists some vector x such that $Ax = b$. Therefore the system $Ax = b$ has a solution.

Thus,

$$Ax = b \text{ has a solution} \iff b \in R(A).$$

Question 10: Prove or give a counterexample to the following statement:

If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

Xingjian's Solution:

The statement is true.

Let A be the coefficient matrix of a system of m linear equations in n unknowns, and suppose that $\text{rank}(A) = m$. Since A has m rows, this means that the rows of A are linearly independent, so A has a pivot in every row.

Now consider the augmented matrix $(A \mid b)$. Appending the column b does not change the fact that all m rows of A already contain pivots. Hence the augmented matrix cannot have a row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & d \end{array}\right], \qquad d \neq 0,$$

because such a row would force a zero row in the coefficient part and therefore would mean that A does not have rank m.

Equivalently, since A has a pivot in every row, the row-reduced form of A has no zero rows. Therefore the system cannot be inconsistent. Hence for every right-hand side b, the system

$$Ax = b$$

has at least one solution.

So the statement is true.

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