Question 1: Label the following statements as true or false.

(a). The rank of a matrix is equal to the number of its nonzero columns.

(b). The product of two matrices always has rank equal to the lesser of the ranks of the two matrices.

(c). The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.

(d). Elementary row operations preserve rank.

(e). Elementary column operations do not necessarily preserve rank.

(f). The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

(g). The inverse of a matrix can be computed exclusively by means of elementary row operations.

(h). The rank of an $n \times n$ matrix is at most n.

(i). An $n \times n$ matrix having rank n is invertible.

Xingjian's Solution:

(a) **False.** The rank is the number of linearly independent columns, not simply the number of nonzero columns. A matrix may have nonzero columns that are multiples of each other.

(b) **False.** In general, $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$, but equality does not always hold. For example, it is possible that $AB = 0$, in which case $\text{rank}(AB) = 0$.

(c) **True.** A matrix has rank 0 exactly when all its rows and all its columns are zero, so it must be the zero matrix.

(d) **True.** Elementary row operations do not change the row space dimension, so they preserve rank.

(e) **False.** Elementary column operations also preserve rank, since they do not change the dimension of the column space.

(f) **True.** By definition, the rank of a matrix is the dimension of its row space, so it equals the maximum number of linearly independent rows.

(g) **True.** If a matrix A is invertible, then by applying elementary row operations to $(A \,|\, I)$, one can reduce A to I, and the same row operations transform I into A^{-1}.

(h) **True.** An $n \times n$ matrix can have at most n linearly independent rows or columns, so its rank is at most n.

(i) **True.** If an $n \times n$ matrix has rank n, then its rows (and columns) are linearly independent, so the matrix is invertible.

Question 2: Find the rank of the following matrices.

(a). $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Xingjian's Solution:
Subtract row 1 from row 3:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are 2 nonzero rows, so $\operatorname{rank}(A) = 2$.

(b). $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Xingjian's Solution:
Perform

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 - R_1,$$

to get

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

There are 3 nonzero rows, so $\operatorname{rank}(A) = 3$.

(c). $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$

Xingjian's Solution:
Perform

$$R_2 \leftarrow R_2 - R_1,$$

to get

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}.$$

There are 2 nonzero rows, so $\operatorname{rank}(A) = 2$.

(d). $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix}$

Xingjian's Solution:
Since $R_2 = 2R_1$, we have

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

There is only 1 nonzero row, so $\operatorname{rank}(A) = 1$.

(e). $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

Xingjian's Solution:
Use row 1 as pivot:

$$R_2 \leftarrow R_2 - R_1, \qquad R_4 \leftarrow R_4 - R_1,$$

giving

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & -2 & -3 & -1 & -1 \end{pmatrix}.$$

Now do

$$R_3 \leftarrow R_3 - R_2, \qquad R_4 \leftarrow R_4 + R_2,$$

to obtain

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & -1 & 0 \end{pmatrix}.$$

There are 3 nonzero rows, so $\text{rank}(A) = 3$.

(f). $\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix}$

Xingjian's Solution:

Notice first that column 2 is 2 times column 1, so they are dependent.

Now row reduce:

$$R_2 \leftarrow R_2 - 2R_1, \quad R_3 \leftarrow R_3 - 3R_1, \quad R_4 \leftarrow R_4 + 4R_1,$$

giving

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 1 & 5 \end{pmatrix}.$$

Next,

$$R_3 \leftarrow R_3 - 2R_2, \qquad R_4 \leftarrow R_4 - R_2,$$

so

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

Finally,

$$R_4 \leftarrow 2R_4 - 7R_3,$$

which gives

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are 3 nonzero rows, so $\operatorname{rank}(A) = 3$.

(g). $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

Xingjian's Solution:

Observe that

$$R_2 = 2R_1, \qquad R_3 = R_1.$$

So perform

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 - R_1, \qquad R_4 \leftarrow R_4 - R_1,$$

to get

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

There are 2 nonzero rows, so $\operatorname{rank}(A) = 2$.

Question 3: Prove that for any $m \times n$ matrix A, $\operatorname{rank}(A) = 0$ if and only if A is the zero matrix.

Xingjian's Solution:

Proof. We prove both directions.

($\Rightarrow$) Assume $\operatorname{rank}(A) = 0$. Recall that the rank of a matrix is the maximum number of linearly independent rows (equivalently, columns). If $\operatorname{rank}(A) = 0$, then there are no nonzero linearly independent rows. Hence every row of A must be the zero row. Therefore every entry of A is 0, so A is the zero matrix.

($\Leftarrow$) Assume A is the zero matrix. Then every row and every column of A is zero, so there are no nonzero linearly independent rows or columns. Hence the dimension of the row space and column space is 0. Therefore

$$\operatorname{rank}(A) = 0.$$

Thus,

$$\operatorname{rank}(A) = 0 \quad \Longleftrightarrow \quad A \text{ is the zero matrix.}$$

□

Question 5: For each of the following matrices, compute the rank and the inverse if it exists.

(a). $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

Xingjian's Solution:

To find the rank and inverse, row reduce $(A \,|\, I)$:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right).$$

First eliminate the entry below the pivot: $R_2 \leftarrow R_2 - R_1$ so

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right).$$

Now make the second pivot equal to 1: $R_2 \leftarrow -R_2$ to get

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right).$$

Eliminate the entry above that pivot: $R_1 \leftarrow R_1 - 2R_2$ and obtain

$$\left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right).$$

Thus the left side has become I_2, so A is invertible. Hence

$$\operatorname{rank}(A) = 2, \qquad A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

(b). $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Xingjian's Solution:

We row reduce $(A \,|\, I)$:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right).$$

Eliminate the entry below the first pivot: $R_2 \leftarrow R_2 - 2R_1$ to get

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right).$$

The left side has only one pivot, so $\operatorname{rank}(A) = 1$. Since the left side cannot be reduced to I_2, the matrix is not invertible. Therefore

$$A^{-1} \text{ does not exist.}$$

(c). $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}$

Xingjian's Solution:

We row reduce $(A \,|\, I)$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right).$$

Use the first row to eliminate the entries below the first pivot:

$$R_2 \leftarrow R_2 - R_1, \qquad R_3 \leftarrow R_3 - 2R_1.$$

Then

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right).$$

Now eliminate the entry below the second pivot:

$$R_3 \leftarrow R_3 + R_2,$$

so

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right).$$

Thus the left side has only 2 pivots, so $\text{rank}(A) = 2$. Since the left side cannot be reduced to I_3, A is not invertible. Hence A^{-1} does not exist.

(d). $\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}$

Xingjian's Solution:

We row reduce $(A \,|\, I)$:

$$\left(\begin{array}{ccc|ccc} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right).$$

Swap rows 1 and 2:

$$R_1 \leftrightarrow R_2$$

to get

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right).$$

Eliminate the entry below the first pivot:

$$R_3 \leftarrow R_3 - 2R_1$$

so

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & -2 & 1 \end{array}\right).$$

Make the second pivot equal to 1:

$$R_2 \leftarrow -\frac{1}{2}R_2$$

to obtain

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & 0 & 0 \\ 0 & 2 & -3 & 0 & -2 & 1 \end{array}\right).$$

Use the second pivot to eliminate the other entries in column 2:

$$R_1 \leftarrow R_1 - R_2, \qquad R_3 \leftarrow R_3 - 2R_2.$$

Then

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right).$$

Eliminate the entries above the third pivot:

$$R_1 \leftarrow R_1 - R_3, \qquad R_2 \leftarrow R_2 + 2R_3.$$

Hence

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 3 & -1 \\ 0 & 1 & 0 & \frac{3}{2} & -4 & 2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right).$$

Thus the left side is I_3, so A is invertible. Therefore

$$\operatorname{rank}(A) = 3, \qquad A^{-1} = \begin{pmatrix} -\frac{1}{2} & 3 & -1 \\ \frac{3}{2} & -4 & 2 \\ 1 & -2 & 1 \end{pmatrix}.$$

(e). $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

Xingjian's Solution:

We row reduce $(A \,|\, I)$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

Use the first row to eliminate the other entries in column 1:

$$R_2 \leftarrow R_2 + R_1, \qquad R_3 \leftarrow R_3 - R_1.$$

Then

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right).$$

Make the second pivot equal to 1:

$$R_2 \leftarrow \frac{1}{3} R_2$$

so

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right).$$

Eliminate the other entries in column 2:

$$R_1 \leftarrow R_1 - 2R_2, \qquad R_3 \leftarrow R_3 + 2R_2.$$

This gives

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 2 & -\frac{1}{3} & \frac{2}{3} & 1 \end{array}\right).$$

Make the third pivot equal to 1:

$$R_3 \leftarrow \frac{1}{2}R_3$$

to obtain

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right).$$

Eliminate the entries above the third pivot:

$$R_1 \leftarrow R_1 + R_3, \qquad R_2 \leftarrow R_2 - R_3.$$

Hence

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right).$$

Thus the left side is I_3, so A is invertible. Therefore

$$\operatorname{rank}(A) = 3, \qquad A^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

(f). $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Xingjian's Solution:

We row reduce $(A \,|\, I)$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

Use the first row to eliminate the entries below the first pivot:

$$R_2 \leftarrow R_2 - R_1, \qquad R_3 \leftarrow R_3 - R_1.$$

Then

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right).$$

Make the second pivot equal to 1:

$$R_2 \leftarrow -\frac{1}{2}R_2$$

so

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right).$$

Eliminate the entry below the second pivot:

$$R_3 \leftarrow R_3 + R_2,$$

which gives

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right).$$

Thus the left side has only 2 pivots, so $\operatorname{rank}(A) = 2$. Since the left side cannot be reduced to I_3, the matrix is not invertible. Therefore A^{-1} does not exist.

(g). $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{pmatrix}$

Xingjian's Solution:

We row reduce $(A \,|\, I)$:

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 5 & 5 & 1 & 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right).$$

Use the first row to eliminate the other entries in column 1:

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 + 2R_1, \qquad R_4 \leftarrow R_4 - 3R_1.$$

Then

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 0 & 1 & 0 \\ 0 & -2 & -5 & -3 & -3 & 0 & 0 & 1 \end{array}\right).$$

Use the second row as pivot row:

$$R_3 \leftarrow R_3 - R_2, \qquad R_4 \leftarrow R_4 + 2R_2, \qquad R_1 \leftarrow R_1 - 2R_2.$$

So we get

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -5 & -2 & 5 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 4 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -7 & 2 & 0 & 1 \end{array}\right).$$

Make the third pivot equal to 1:

$$R_3 \leftarrow -R_3$$

to obtain

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -5 & -2 & 5 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -7 & 2 & 0 & 1 \end{array}\right).$$

Eliminate the other entries in column 3:

$$R_4 \leftarrow R_4 - R_3, \qquad R_2 \leftarrow R_2 - 3R_3, \qquad R_1 \leftarrow R_1 + 5R_3.$$

Then

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -12 & -15 & 3 & -5 & 0 \\ 0 & 1 & 0 & 7 & 10 & -2 & 3 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right).$$

Use the fourth pivot to eliminate the entries above it:

$$R_1 \leftarrow R_1 + 12R_4, \qquad R_2 \leftarrow R_2 - 7R_4, \qquad R_3 \leftarrow R_3 + 2R_4.$$

Hence

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -51 & 15 & 7 & 12 \\ 0 & 1 & 0 & 0 & 31 & -9 & -4 & -7 \\ 0 & 0 & 1 & 0 & -10 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right).$$

Thus the left side is I_4, so A is invertible. Therefore

$$\operatorname{rank}(A) = 4, \qquad A^{-1} = \begin{pmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix}.$$

(h). $\begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$

Xingjian's Solution:

We row reduce $(A \,|\, I)$:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right).$$

Use the first row to eliminate the entries below the first pivot:

$$R_2 \leftarrow R_2 - R_1, \qquad R_3 \leftarrow R_3 - 2R_1.$$

Then

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right).$$

Use the second row as pivot row:

$$R_4 \leftarrow R_4 + R_2$$

so

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 & 0 & 1 \end{array}\right).$$

Swap rows 3 and 4:

$$R_3 \leftrightarrow R_4$$

to get

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 & 0 & 1 \\ 0 & 0 & 3 & -2 & -2 & 0 & 1 & 0 \end{array}\right).$$

Eliminate the other entries in column 3:

$$R_1 \leftarrow R_1 + R_3, \qquad R_4 \leftarrow R_4 - 3R_3.$$

Then

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 & 1 & -3 & 1 & -3 \end{array}\right).$$

Make the fourth pivot equal to 1:

$$R_4 \leftarrow \frac{1}{4} R_4$$

to obtain

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & -\frac{3}{4} \end{array}\right).$$

Eliminate the entries above the fourth pivot:

$$R_1 \leftarrow R_1 + R_4, \qquad R_2 \leftarrow R_2 - R_4, \qquad R_3 \leftarrow R_3 + 2R_4.$$

Hence

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & -\frac{5}{4} & \frac{7}{4} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & -\frac{3}{4} \end{array}\right).$$

Thus the left side is I_4, so A is invertible. Therefore

$$\operatorname{rank}(A) = 4, \qquad A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{5}{4} & \frac{7}{4} & -\frac{1}{4} & \frac{3}{4} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & -\frac{3}{4} \end{pmatrix}.$$

Question 6: For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists.

(a). $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, defined by $T(f(x)) = f''(x) + 2f'(x) - f(x)$.

Xingjian's Solution:

Consider $f(x) = a + bx + cx^2 \in P_2(\mathbb{R})$. Then $f'(x) = b + 2cx$, $f''(x) = 2c$. Hence

$$T(f(x)) = f''(x) + 2f'(x) - f(x) = 2c + 2(b + 2cx) - (a + bx + cx^2).$$

So $T(f(x)) = (-a + 2b + 2c) + (-b + 4c)x - cx^2$.

To determine whether T is invertible, suppose $T(f(x)) = 0$. Then

$$-a + 2b + 2c = 0, \qquad -b + 4c = 0, \qquad -c = 0.$$

From $-c = 0$, we get $c = 0$. Then $-b + 4c = 0$ gives $b = 0$, and finally $-a + 2b + 2c = 0$ gives $a = 0$. Thus

$$f(x) = 0.$$

So $\ker(T) = \{0\}$. Since $P_2(\mathbb{R})$ is finite-dimensional and the domain and codomain both have dimension 3, T is invertible.

Now let

$$T(f(x)) = p + qx + rx^2.$$

Then

$$-a + 2b + 2c = p, \qquad -b + 4c = q, \qquad -c = r.$$

From the third equation, $c = -r$. From the second equation,

$$-b + 4(-r) = q \quad \Longrightarrow \quad b = -q - 4r.$$

From the first equation,

$$-a + 2(-q - 4r) + 2(-r) = p$$

so

$$-a - 2q - 10r = p,$$

hence

$$a = -p - 2q - 10r.$$

Therefore

$$T^{-1}(p + qx + rx^2) = (-p - 2q - 10r) + (-q - 4r)x - rx^2.$$

So T is invertible, and

$$\boxed{T^{-1}(p + qx + rx^2) = (-p - 2q - 10r) + (-q - 4r)x - rx^2.}$$

(b). $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, defined by $T(f(x)) = (x + 1)f'(x)$.
Xingjian's Solution:
Consider $f(x) = a + bx + cx^2 \in P_2(\mathbb{R})$. Then $f'(x) = b + 2cx$. Therefore

$$T(f(x)) = (x + 1)f'(x) = (x + 1)(b + 2cx) = b + (b + 2c)x + 2cx^2.$$

To test invertibility, we check whether T is one-to-one. Consider any constant polynomial $f(x) = a, a \neq 0$. Then $f'(x) = 0$, so $T(f(x)) = (x + 1) \cdot 0 = 0$. Thus a nonzero polynomial is mapped to zero, so

$$\ker(T) \neq \{0\}.$$

Therefore T is not one-to-one, hence not invertible. So T^{-1} does not exist.

$$\boxed{T \text{ is not invertible, and } T^{-1} \text{ does not exist.}}$$

(c). $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(a_1, a_2, a_3) = (a_1 + 2a_2 + a_3,\ -a_1 + a_2 + 2a_3,\ a_1 + a_3)$.
Xingjian's Solution:
Let $T(a_1, a_2, a_3) = (b_1, b_2, b_3)$. Then we must solve

$$a_1 + 2a_2 + a_3 = b_1,$$

$$-a_1 + a_2 + 2a_3 = b_2,$$

$$a_1 + a_3 = b_3.$$

We solve for a_1, a_2, a_3 in terms of b_1, b_2, b_3.
From the third equation, $a_1 = b_3 - a_3$. Substitute into the first equation:

$$(b_3 - a_3) + 2a_2 + a_3 = b_1 \quad \Longrightarrow \quad b_3 + 2a_2 = b_1.$$

Hence $a_2 = \frac{b_1 - b_3}{2}$.

Now substitute $a_1 = b_3 - a_3$ into the second equation: $-(b_3 - a_3) + a_2 + 2a_3 = b_2$, so $-b_3 + a_2 + 3a_3 = b_2$. Thus $3a_3 = b_2 + b_3 - a_2$. Substitute $a_2 = \dfrac{b_1 - b_3}{2}$:

$$3a_3 = b_2 + b_3 - \frac{b_1 - b_3}{2} = \frac{2b_2 + 2b_3 - b_1 + b_3}{2} = \frac{-b_1 + 2b_2 + 3b_3}{2}.$$

Therefore

$$a_3 = \frac{-b_1 + 2b_2 + 3b_3}{6}.$$

Then

$$a_1 = b_3 - a_3 = b_3 - \frac{-b_1 + 2b_2 + 3b_3}{6} = \frac{b_1 - 2b_2 + 3b_3}{6}.$$

So

$$a_1 = \frac{b_1 - 2b_2 + 3b_3}{6}, \qquad a_2 = \frac{b_1 - b_3}{2}, \qquad a_3 = \frac{-b_1 + 2b_2 + 3b_3}{6}.$$

Since for every $(b_1, b_2, b_3) \in \mathbb{R}^3$ there is a unique solution, T is invertible, and

$$\boxed{T^{-1}(b_1, b_2, b_3) = \left(\frac{b_1 - 2b_2 + 3b_3}{6}, \frac{b_1 - b_3}{2}, \frac{-b_1 + 2b_2 + 3b_3}{6}\right).}$$

(d). $T : \mathbb{R}^3 \to P_2(\mathbb{R})$, defined by $T(a_1, a_2, a_3) = (a_1 + a_2 + a_3) + (a_1 - a_2 + a_3)x + a_1x^2$.
Xingjian's Solution:
Let $T(a_1, a_2, a_3) = p + qx + rx^2$. Then comparing coefficients gives

$$a_1 + a_2 + a_3 = p,$$

$$a_1 - a_2 + a_3 = q,$$

$$a_1 = r.$$

From the third equation, $a_1 = r$.
Subtract the second equation from the first:

$$(a_1 + a_2 + a_3) - (a_1 - a_2 + a_3) = p - q,$$

so $a_2 = \frac{p-q}{2}$.
Now use

$$a_1 + a_2 + a_3 = p.$$

Substitute $a_1 = r$ and $a_2 = \dfrac{p-q}{2}$:

$$r + \frac{p-q}{2} + a_3 = p.$$

Thus

$$a_3 = p - r - \frac{p-q}{2} = \frac{2p - 2r - p + q}{2} = \frac{p+q}{2} - r.$$

Therefore

$$a_1 = r, \qquad a_2 = \frac{p-q}{2}, \qquad a_3 = \frac{p+q}{2} - r.$$

So T is invertible, and

$$\boxed{T^{-1}(p + qx + rx^2) = \left(r, \frac{p-q}{2}, \frac{p+q}{2} - r\right).}$$

(e). $T : P_2(\mathbb{R}) \to \mathbb{R}^3$, defined by $T(f(x)) = (f(-1), f(0), f(1))$.

Xingjian's Solution:

Let $f(x) = a + bx + cx^2$. Then $f(-1) = a - b + c, f(0) = a, f(1) = a + b + c$. So if $T(f(x)) = (u, v, w)$, then

$$a - b + c = u,$$

$$a = v,$$

$$a + b + c = w.$$

From the second equation, $a = v$.

Then the first and third equations become

$$v - b + c = u,$$

$$v + b + c = w.$$

Subtract the first from the third:

$$(v + b + c) - (v - b + c) = w - u,$$

so $b = \frac{w-u}{2}$.

Now add the first and third equations:

$$(v - b + c) + (v + b + c) = u + w,$$

which gives

$$2v + 2c = u + w.$$

So

$$2c = u + w - 2v, \qquad c = \frac{u - 2v + w}{2}.$$

Thus

$$a = v, \qquad b = \frac{w - u}{2}, \qquad c = \frac{u - 2v + w}{2}.$$

Therefore

$$f(x) = v + \frac{w - u}{2}x + \frac{u - 2v + w}{2}x^2.$$

So T is invertible, and

$$\boxed{T^{-1}(u, v, w) = v + \frac{w - u}{2}x + \frac{u - 2v + w}{2}x^2.}$$

(f). $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$, defined by $T(A) = (\mathrm{tr}(A), \mathrm{tr}(A^t), \mathrm{tr}(EA), \mathrm{tr}(AE))$, where $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Xingjian's Solution:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\mathrm{tr}(A) = a + d$. Also, $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, so $\mathrm{tr}(A^t) = a + d$.

Now

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad EA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix},$$

hence

$$\operatorname{tr}(EA) = b + c.$$

Also

$$AE = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix},$$

so

$$\operatorname{tr}(AE) = b + c.$$

Therefore

$$T(A) = (a + d,\ a + d,\ b + c,\ b + c).$$

So every vector in the range of T has the form

$$(x, x, y, y).$$

In particular, vectors such as $(1, 0, 0, 0)$ are not in the range. Hence T is not onto $\mathbb{R}^4$.

Also, many different matrices have the same image. For example,

$$T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = (1, 1, 0, 0) = T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

So T is not one-to-one.

Therefore T is not invertible, and T^{-1} does not exist.

$$\boxed{T \text{ is not invertible, and } T^{-1} \text{ does not exist.}}$$

Question 8: Let A be an $m \times n$ matrix. Prove that if c is any nonzero scalar, then

$$\operatorname{rank}(cA) = \operatorname{rank}(A).$$

Xingjian's Solution:

Proof. Observe that multiplying a matrix by a nonzero scalar multiplies every row of the matrix by that same nonzero scalar. This does not change whether a collection of rows is linearly independent.

More precisely, suppose the rows of A are $r_1, \ldots, r_m$. Then the rows of cA are $cr_1, \ldots, cr_m$. A set of rows $r_{i_1}, \ldots, r_{i_k}$ is linearly independent if and only if the set $cr_{i_1}, \ldots, cr_{i_k}$ is linearly independent, because

$$\alpha_1(cr_{i_1}) + \cdots + \alpha_k(cr_{i_k}) = 0$$

if and only if

$$c(\alpha_1 r_{i_1} + \cdots + \alpha_k r_{i_k}) = 0.$$

Since $c \neq 0$, this holds if and only if

$$\alpha_1 r_{i_1} + \cdots + \alpha_k r_{i_k} = 0.$$

Thus the maximum number of linearly independent rows is the same for A and cA.

Therefore

$$\operatorname{rank}(cA) = \operatorname{rank}(A).$$

Alternatively, multiplying every row by a nonzero scalar is a type 2 elementary row operation, and elementary row operations preserve rank. Hence the conclusion follows immediately. □

Question 14: Let $T, U : V \to W$ be linear transformations.
(a). Prove that $\mathrm{R}(T+U) \subseteq \mathrm{R}(T) + \mathrm{R}(U)$.
(b). Prove that if W is finite-dimensional, then

$$\operatorname{rank}(T+U) \leq \operatorname{rank}(T) + \operatorname{rank}(U).$$

(c). Deduce that $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$ for any $m \times n$ matrices A and B.
Xingjian's Solution:

Proof For (a). Let $y \in \mathrm{R}(T+U)$. Then by definition of range, there exists $v \in V$ such that $y = (T+U)(v)$. Since $T+U$ is defined pointwise, $(T+U)(v) = T(v) + U(v)$. Now $T(v) \in \mathrm{R}(T)$ and $U(v) \in \mathrm{R}(U)$. Therefore $y = T(v) + U(v) \in \mathrm{R}(T) + \mathrm{R}(U)$. Hence $\mathrm{R}(T+U) \subseteq \mathrm{R}(T) + \mathrm{R}(U)$. □

Proof For (b). By part (a), $\mathrm{R}(T+U) \subseteq \mathrm{R}(T) + \mathrm{R}(U)$. Since W is finite-dimensional, we may take dimensions:

$$\dim \mathrm{R}(T+U) \leq \dim \big(\mathrm{R}(T) + \mathrm{R}(U)\big).$$

That is,

$$\operatorname{rank}(T+U) \leq \dim \big(\mathrm{R}(T) + \mathrm{R}(U)\big).$$

Now for subspaces $X, Y \subseteq W$, we know

$$\dim(X+Y) = \dim(X) + \dim(Y) - \dim(X \cap Y) \leq \dim(X) + \dim(Y).$$

Apply this with $X = \mathrm{R}(T), Y = \mathrm{R}(U)$. Then

$$\dim \big(\mathrm{R}(T) + \mathrm{R}(U)\big) \leq \dim \mathrm{R}(T) + \dim \mathrm{R}(U).$$

Hence

$$\operatorname{rank}(T+U) \leq \operatorname{rank}(T) + \operatorname{rank}(U).$$

□

Proof For (c). Let A and B be $m \times n$ matrices. Define linear transformations $T_A, T_B : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(x) = Ax, T_B(x) = Bx$. Then $(T_A + T_B)(x) = Ax + Bx = (A+B)x$, so the matrix of $T_A + T_B$ is $A + B$.

Now

$$\operatorname{rank}(A) = \operatorname{rank}(T_A), \qquad \operatorname{rank}(B) = \operatorname{rank}(T_B), \qquad \operatorname{rank}(A+B) = \operatorname{rank}(T_A + T_B).$$

By part (b),

$$\operatorname{rank}(T_A + T_B) \leq \operatorname{rank}(T_A) + \operatorname{rank}(T_B).$$

Therefore

$$\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

□

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