Question 1: Label the following statements as true or false.

(a). An elementary matrix is always square.

(b). The only entries of an elementary matrix are zeros and ones.

(c). The $n \times n$ identity matrix is an elementary matrix.

(d). The product of two $n \times n$ elementary matrices is an elementary matrix.

(e). The inverse of an elementary matrix is an elementary matrix.

(f). The sum of two $n \times n$ matrices is an elementary matrix.

(g). The transpose of an elementary matrix is an elementary matrix.

(h) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.

(i). If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

Xingjian's Solution:

Recall that an elementary matrix is obtained by performing exactly one elementary row operation on an identity matrix.

(a) **True.** Since an elementary matrix is obtained from an identity matrix, and every identity matrix is square, every elementary matrix is square.

(b) **False.** This is not always true. For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

is an elementary matrix obtained by multiplying the second row of I_2 by 3, and it contains an entry 3, not just 0 and 1.

(c) **False.** An elementary matrix is obtained by performing *one* elementary row operation on the identity matrix. If no operation is performed, we still have the identity matrix, but that is not considered an elementary matrix in the usual definition.

(d) **False.** The product of two elementary matrices corresponds to performing two row operations, so in general it is not a single elementary matrix. For example,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

and this cannot be obtained from I_2 by just one elementary row operation.

(e) **True.** The inverse of each type of elementary row operation is again an elementary row operation: swapping two rows is its own inverse; multiplying a row by $c \neq 0$ is inverted by multiplying the same row by $1/c$; adding c times one row to another is inverted by adding $-c$ times that row. Hence the inverse of an elementary matrix is elementary.

(f) **False.** The sum of two matrices need not be elementary. For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix},$$

which is not obtained from I_2 by one elementary row operation.

(g) **True.** The transpose of each elementary matrix is again elementary: swapping rows i, j gives a symmetric matrix; scaling row i gives a diagonal matrix, whose transpose is itself; adding c times row j to row i transposes to adding c times row i to row j, which is also elementary.

(h) **False.** A row operation changes rows, while a column operation changes columns. In general, doing a row operation on A does not produce the same matrix as doing a single column operation on A. For example, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Swap the two rows to get

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

But no single column operation on A produces B, because column operations cannot move the nonzero entry from the first row to the second row in this way.

(i) **True.** Every elementary row operation is reversible by another elementary row operation: swap back the two rows; multiply by the reciprocal scalar; add the negative multiple.

So if B is obtained from A by an elementary row operation, then A is obtained from B by an elementary row operation.

Question 2: Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}.$$

Find an elementary operation that transforms A into B and an elementary operation that transforms B into C. By means of several additional operations, transform C into I_3.

Xingjian's Solution:

First, transform A into B.

Observe that

$$R_2 - R_1 = (1, 0, 1) - (1, 2, 3) = (0, -2, -2),$$

but this is not the second row of B. Instead notice that subtracting the second column effect suggests:

$$C_2 = C_2 - 2C_1$$

would not work since the problem asks for an elementary operation from A to B, and checking row operations:

$$R_1 \mapsto R_1 - 2R_2$$

gives

$$(1,2,3) - 2(1,0,1) = (-1,2,1),$$

so that does not work.

Instead, compare rows directly:

$$(1,0,3) = (1,2,3) - 2(0,1,0),$$

which is not available from a row operation. So let us compare columns:

$$\text{col}_2(B) = \text{col}_2(A) - 2\text{col}_1(A).$$

Indeed,

$$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix}.$$

Thus $A \to B$ is obtained by the elementary *column* operation

$$C_2 \leftarrow C_2 - 2C_1.$$

Next, transform B into C.

Comparing B and C, the first row and third row stay the same, and

$$(0,-2,-2) = (1,-2,1) - (1,0,3).$$

Hence the operation is

$$R_2 \leftarrow R_2 - R_1.$$

Now transform C into I_3 by elementary row operations.

Start with

$$C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}.$$

$$R_3 \leftarrow R_3 - R_1$$

gives

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{pmatrix}.$$

$$R_2 \leftarrow -\frac{1}{2} R_2$$

gives

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{pmatrix}.$$

$$R_3 \leftarrow R_3 + 3R_2$$

gives

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$R_2 \leftarrow R_2 - R_3$$

gives

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$R_1 \leftarrow R_1 - 3R_3$$

gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

Therefore:

$$A \xrightarrow{C_2 \leftarrow C_2 - 2C_1} B \xrightarrow{R_2 \leftarrow R_2 - R_1} C \xrightarrow{\text{row operations}} I_3.$$

Question 3: Use the proof of Theorem 3.2 to obtain the inverse of each of the following elementary matrices.

$$1.\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad 2.\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad 3.\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Xingjian's Solution:

We find the inverse of each elementary matrix by reversing the corresponding elementary operation.

1. This matrix E is obtained by interchanging rows 1 and 3 of I_3. Swapping the same two rows again restores the identity, so

$$E^{-1} = E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

2. This is obtained by multiplying row 2 of I_3 by 3. The inverse operation is multiplying row 2 by 1/3. Hence
$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. This is obtained by adding -2 times row 1 to row 3. The inverse operation is adding 2 times row 1 to row 3. Therefore
$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Question 5: Prove that E is an elementary matrix if and only if E^t is.

Xingjian's Solution:

Suppose E is an elementary matrix. Then E is obtained from an identity matrix I_n by one elementary row operation.

Proof. We consider the three types of elementary row operations.

Type 1: Interchange two rows of I_n. The resulting matrix is symmetric, so its transpose is itself. Hence E^t is again elementary.

Type 2: Multiply one row of I_n by a nonzero scalar c. Then E is a diagonal matrix with one diagonal entry equal to c and all others equal to 1. Such a matrix is equal to its transpose, so E^t is elementary.

Type 3: Add c times row j to row i. Then E has a single off-diagonal entry c in position (i, j), together with 1's on the diagonal. Its transpose has the same form, but now the off-diagonal entry c is in position (j, i). Thus E^t is the elementary matrix corresponding to adding c times row i to row j. Hence E^t is elementary.

So in every case, if E is elementary, then E^t is elementary.

Conversely, if E^t is elementary, then by the same argument $(E^t)^t = E$ is elementary.

Therefore,
$$E \text{ is elementary} \iff E^t \text{ is elementary.}$$

□

Question 6: Let A be an $m \times n$ matrix. Prove that if B can be obtained from A by an elementary row (column) operation, then B^t can be obtained from A^t by the corresponding elementary column (row) operation.

Xingjian's Solution:

Proof. Assume B is obtained from A by an elementary row operation. Then there exists an elementary matrix E such that
$$B = EA.$$
Taking transposes, we get
$$B^t = (EA)^t = A^t E^t.$$

By Question 5, E^t is also an elementary matrix. Right multiplication by an elementary matrix corresponds to an elementary column operation. Hence B^t is obtained from A^t by the corresponding elementary column operation.

The proof for column operations is analogous. If B is obtained from A by an elementary column operation, then

$$B = AE$$

for some elementary matrix E. Taking transposes gives

$$B^t = (AE)^t = E^t A^t.$$

Since E^t is elementary, left multiplication by E^t corresponds to an elementary row operation. Thus B^t is obtained from A^t by the corresponding elementary row operation.

Therefore, if B is obtained from A by an elementary row (column) operation, then B^t is obtained from A^t by the corresponding elementary column (row) operation. □

Question 10: Prove that any elementary row (column) operation of type 2 can be obtained by dividing some row (column) by a nonzero scalar.

Xingjian's Solution:

Proof. A type 2 elementary row operation means multiplying some row by a nonzero scalar c.

But multiplying a row by $c \neq 0$ is the same as dividing that row by $1/c$, which is also nonzero. Indeed,

$$cR_i = \frac{1}{1/c} R_i.$$

So every type 2 row operation can be described as dividing a row by a nonzero scalar.

Conversely, dividing row R_i by a nonzero scalar d is the same as multiplying R_i by $1/d$, which is again a type 2 row operation.

The same argument works for columns.

Hence any elementary row (column) operation of type 2 can be obtained by dividing some row (column) by a nonzero scalar. □

Question 11: Prove that any elementary row (column) operation of type 3 can be obtained by subtracting a multiple of some row (column) from another row (column).

Xingjian's Solution:

Proof. Recall a type 3 elementary row operation has the form $R_i \leftarrow R_i + cR_j \qquad (i \neq j)$, where c is any scalar.

But addition of cR_j can always be rewritten as subtraction of $(-c)R_j$: $R_i + cR_j = R_i - (-c)R_j$. Thus every type 3 row operation can be expressed as subtracting a multiple of one row from another.

Conversely, an operation of the form $R_i \leftarrow R_i - cR_j$ is the same as $R_i \leftarrow R_i + (-c)R_j$, which is a type 3 row operation.

The same reasoning applies to columns:

$$C_i \leftarrow C_i + cC_j \quad \text{is the same as} \quad C_i \leftarrow C_i - (-c)C_j.$$

Therefore any elementary row (column) operation of type 3 can be obtained by subtracting a multiple of some row (column) from another row (column). □

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