Question 8: Prove the following generalization of Theorem 2.23. Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W. Let β and β' be ordered bases for V, and let γ and γ' be ordered bases for W. Then

$$[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma}Q,$$

where Q is the matrix that changes β'-coordinates into β-coordinates and P is the matrix that changes γ'-coordinates into γ-coordinates.

Xingjian's Solution:

Proof. Let $x \in V$. Since Q is the matrix that changes β'-coordinates into β-coordinates, we have $[x]_\beta = Q[x]_{\beta'}$.

Applying T, and using the definition of matrix representation with respect to β and γ, we obtain $[Tx]_\gamma = [T]_\beta^\gamma[x]_\beta$.

Substituting the previous expression gives $[Tx]_\gamma = [T]_\beta^\gamma Q[x]_{\beta'}$. Since P changes γ'-coordinates into γ-coordinates, we also have

$$[Tx]_\gamma = P[Tx]_{\gamma'}.$$

Thus,

$$P[Tx]_{\gamma'} = [T]_\beta^\gamma Q[x]_{\beta'}.$$

Multiplying both sides by P^{-1} yields

$$[Tx]_{\gamma'} = P^{-1}[T]_\beta^\gamma Q[x]_{\beta'}.$$

By definition of $[T]_{\beta'}^{\gamma'}$, we also have

$$[Tx]_{\gamma'} = [T]_{\beta'}^{\gamma'}[x]_{\beta'}.$$

Hence,

$$[T]_{\beta'}^{\gamma'}[x]_{\beta'} = P^{-1}[T]_\beta^\gamma Q[x]_{\beta'}.$$

Since this holds for all $[x]_{\beta'}$, we conclude

$$[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q.$$

□

Question 12: Prove the corollary to Theorem 2.23.

Xingjian's Solution:

Proof. Apply the result of Question 8 in the special case where $W = V$, $\gamma = \beta$, and $\gamma' = \beta'$. Then the matrix P coincides with the matrix Q that changes β'-coordinates into β-coordinates. Hence,

$$[T]_{\beta'}^{\beta'} = Q^{-1}[T]_\beta^\beta Q.$$

Using the standard notation

$$[T]_{\beta'} = [T]_{\beta'}^{\beta'}, \quad [T]_\beta = [T]_\beta^\beta,$$

we obtain

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q.$$

□

Question 14: Prove the converse of Exercise 8: If A and B are each $m \times n$ matrices with entries from a field F, and if there exist invertible $m \times m$ and $n \times n$ matrices P and Q, respectively, such that $B = P^{-1}AQ$, then there exist an n-dimensional vector space V (both over F), ordered bases β and β' for V, ordered bases γ and γ' for W, and a linear transformation $T : V \to W$ such that

$$A = [T]_{\beta}^{\gamma} \quad \text{and} \quad B = [T]_{\beta'}^{\gamma'}.$$

Hints: Let $V = F^n$, $W = F^m$, $T = L_A$, and let β and γ be the standard ordered bases for F^n and F^m, respectively. Now apply the results of Exercise 13 to obtain ordered bases β' and γ' from β and γ via Q and P, respectively.

Xingjian's Solution:

Proof. Let $V = F^n$ and $W = F^m$, and let β and γ be the standard ordered bases of F^n and F^m, respectively.

Define $T = L_A : F^n \to F^m$ by

$$T(x) = Ax.$$

Then, with respect to the standard bases,

$$[T]_{\beta}^{\gamma} = A.$$

Since Q is invertible, there exists an ordered basis β' of V such that Q is the matrix that changes β'-coordinates into β-coordinates. Similarly, since P is invertible, there exists an ordered basis γ' of W such that P changes γ'-coordinates into γ-coordinates.

Applying the result of Question 8, we obtain

$$[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma}Q.$$

Substituting $[T]_{\beta}^{\gamma} = A$ gives

$$[T]_{\beta'}^{\gamma'} = P^{-1}AQ.$$

By hypothesis, $P^{-1}AQ = B$, so

$$[T]_{\beta'}^{\gamma'} = B.$$

Thus,

$$A = [T]_{\beta}^{\gamma}, \quad B = [T]_{\beta'}^{\gamma'}.$$

□

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