Question 1: Label the following statements as **true** or **false**.

(a). Suppose that $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x_1', x_2', \ldots, x_n'\}$ are ordered bases for a vector space V and Q is the change of coordinate matrix that changes β-coordinates into β'-coordinates. Then the jth column of Q is $[x_j]_{\beta'}$.

(b). Every change of coordinate matrix is invertible.

(c). Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

(d). The matrices $A, B \in M_{n\times n}(F)$ are called similar if $B = Q^{-1}AQ$ for some $Q \in M_{n\times n}(F)$.

(e). Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and β' for V, $[T]_\beta$ is similar to $[T]_{\beta'}$.

Xingjian's Solution:

(a). **True.** If Q changes β-coordinates into β'-coordinates, then $Q[x]_\beta = [x]_{\beta'}$ for all $x \in V$. In particular, for $x = x_j$, we have $[x_j]_\beta = e_j$, so $Qe_j = [x_j]_{\beta'}$. But Qe_j is exactly the jth column of Q. Hence the jth column of Q is $[x_j]_{\beta'}$.

(b). **True.** A change of coordinate matrix gives a bijection between coordinate vectors relative to two ordered bases. Hence it is invertible.

(c). **True.** Since Q changes β'-coordinates into β-coordinates, Theorem 2.23 gives $[T]_{\beta'} = Q^{-1}[T]_\beta Q$. Multiplying by Q on the left and Q^{-1} on the right yields $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

(d). **False.** The correct definition of similarity is: A and B are similar if $B = Q^{-1}AQ$ for some invertible matrix Q. The statement omits the requirement that Q be invertible.

(e). **True.** By Theorem 2.23, if Q changes β'-coordinates into β-coordinates, then $[T]_{\beta'} = Q^{-1}[T]_\beta Q$. Thus $[T]_\beta$ and $[T]_{\beta'}$ are similar.

Question 2: For each of the following pairs of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

Xingjian comment: For a basis $\beta = \{v_1, v_2\}$ and $\beta' = \{w_1, w_2\}$, the change of coordinate matrix from β'-coordinates into β-coordinates is actually the matrix whose columns are $[w_1]_\beta$, $[w_2]_\beta$.

(a). $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$

Xingjian's Solution:

Since $\beta = \{e_1, e_2\}$ is the standard basis, the coordinates of the vectors in β' are just their usual column vectors. Hence

$$Q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

(b). $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$

Xingjian's Solution:

We write $(0, 10) = c_1(-1, 3) + c_2(2, -1)$. Then $-c_1 + 2c_2 = 0, 3c_1 - c_2 = 10$, which gives $c_1 = 4$, $c_2 = 2$. Thus

$$[(0, 10)]_\beta = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Next write $(5, 0) = d_1(-1, 3) + d_2(2, -1)$. Then $-d_1 + 2d_2 = 5, 3d_1 - d_2 = 0$, which gives

$d_1 = 1,\ d_2 = 3$. Thus

$$[(5,0)]_\beta = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Therefore,

$$Q = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

(c). $\beta = \{(2,5),(-1,-3)\}$ and $\beta' = \{e_1, e_2\}$

Xingjian's Solution:

We express $e_1 = (1,0)$ and $e_2 = (0,1)$ in the basis β.

For e_1, $(1,0) = c_1(2,5) + c_2(-1,-3)$, so $2c_1 - c_2 = 1, 5c_1 - 3c_2 = 0$. This gives $c_1 = 3,\ c_2 = 5$. Hence $[e_1]_\beta = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

For e_2, $(0,1) = d_1(2,5) + d_2(-1,-3)$, so $2d_1 - d_2 = 0, 5d_1 - 3d_2 = 1$. This gives $d_1 = -1,\ d_2 = -2$. Hence $[e_2]_\beta = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

Therefore,

$$Q = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}.$$

(d). $\beta = \{(-4,3),(2,-1)\}$ and $\beta' = \{(2,1),(-4,1)\}$

Xingjian's Solution: First, $(2,1) = c_1(-4,3) + c_2(2,-1)$. Then $-4c_1 + 2c_2 = 2, 3c_1 - c_2 = 1$. So $c_1 = 2,\ c_2 = 5$, and hence $[(2,1)]_\beta = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Next, $(-4,1) = d_1(-4,3) + d_2(2,-1)$. Then $-4d_1 + 2d_2 = -4, 3d_1 - d_2 = 1$. So $d_1 = -1,\ d_2 = -4$, and hence $[(-4,1)]_\beta = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

Therefore,

$$Q = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}.$$

Question 3: For each of the following pairs of ordered bases β and β' for $P_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

Xingjian comment: each part, the change of coordinate matrix from β'-coordinates into β-coordinates is the matrix whose columns are the coordinate vectors of the elements of β' written in the basis β.

(a). $\beta = \{x^2, x, 1\}$ and $\beta' = \{a_2x^2 + a_1x + a_0,\ b_2x^2 + b_1x + b_0,\ c_2x^2 + c_1x + c_0\}$

Xingjian's Solution: Since $\beta = \{x^2, x, 1\}$, we directly read off coefficients:

$$[a_2x^2 + a_1x + a_0]_\beta = \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix}, \quad [b_2x^2 + b_1x + b_0]_\beta = \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}, \quad [c_2x^2 + c_1x + c_0]_\beta = \begin{pmatrix} c_2 \\ c_1 \\ c_0 \end{pmatrix}.$$

Hence

$$Q = \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix}.$$

(b). $\beta = \{1, x, x^2\}$ and $\beta' = \{a_2x^2 + a_1x + a_0,\ b_2x^2 + b_1x + b_0,\ c_2x^2 + c_1x + c_0\}$

Xingjian's Solution: Since $\beta = \{1, x, x^2\}$, the coordinates are now taken in the order constant, linear, quadratic:

$$[a_2x^2 + a_1x + a_0]_\beta = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}, \quad [b_2x^2 + b_1x + b_0]_\beta = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}, \quad [c_2x^2 + c_1x + c_0]_\beta = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}.$$

Thus

$$Q = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

(c). $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$

Xingjian's Solution: We write each polynomial in β' as a linear combination of the basis β.

For 1: $1 = a(2x^2 - x) + b(3x^2 + 1) + cx^2$. Comparing coefficients gives $-a = 0, b = 1, 2a + 3b + c = 0$, so $a = 0,\ b = 1,\ c = -3$. Hence $[1]_\beta = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$.

For x: $x = a(2x^2 - x) + b(3x^2 + 1) + cx^2$. Comparing coefficients gives $-a = 1, b = 0, 2a + 3b + c = 0$, so $a = -1,\ b = 0,\ c = 2$. Hence $[x]_\beta = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$.

For x^2: $x^2 = a(2x^2 - x) + b(3x^2 + 1) + cx^2$. Comparing coefficients gives $-a = 0, b = 0, 2a + 3b + c = 1$, so $a = 0,\ b = 0,\ c = 1$. Hence $[x^2]_\beta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Therefore,

$$Q = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix}.$$

(d). $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and $\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$

Xingjian's Solution: For $x^2 + x + 4$, write $x^2 + x + 4 = a(x^2 - x + 1) + b(x + 1) + c(x^2 + 1)$. Comparing coefficients: $a + c = 1, -a + b = 1, a + b + c = 4$. This gives $a = 2,\ b = 3,\ c = -1$. Hence $[x^2 + x + 4]_\beta = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

For $4x^2 - 3x + 2$, write $4x^2 - 3x + 2 = a(x^2 - x + 1) + b(x + 1) + c(x^2 + 1)$. Comparing coefficients: $a + c = 4, -a + b = -3, a + b + c = 2$. This gives $a = 1,\ b = -2,\ c = 3$. Hence $[4x^2 - 3x + 2]_\beta = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

For $2x^2+3$, write $2x^2+3 = a(x^2-x+1)+b(x+1)+c(x^2+1)$. Comparing coefficients: $a+c=2, -a+b=0, a+b+c=3$. This gives $a=1,\ b=1,\ c=1$. Hence $[2x^2+3]_\beta = \begin{pmatrix}1\\1\\1\end{pmatrix}$.

Therefore, $Q = \begin{pmatrix}2 & 1 & 1\\ 3 & -2 & 1\\ -1 & 3 & 1\end{pmatrix}$.

(e). $\beta = \{x^2-x, x^2+1, x-1\}$ and $\beta' = \{5x^2-2x-3, -2x^2+5x+5, 2x^2-x-3\}$

Xingjian's Solution: For $5x^2-2x-3$, write $5x^2-2x-3 = a(x^2-x)+b(x^2+1)+c(x-1)$. Comparing coefficients: $a+b=5, -a+c=-2, b-c=-3$. This gives $a=5,\ b=0,\ c=3$. Hence $[5x^2-2x-3]_\beta = \begin{pmatrix}5\\0\\3\end{pmatrix}$.

For $-2x^2+5x+5$, write $-2x^2+5x+5 = a(x^2-x)+b(x^2+1)+c(x-1)$. Comparing coefficients: $a+b=-2, -a+c=5, b-c=5$. This gives $a=-6,\ b=4,\ c=-1$. Hence $[-2x^2+5x+5]_\beta = \begin{pmatrix}-6\\4\\-1\end{pmatrix}$.

For $2x^2-x-3$, write $2x^2-x-3 = a(x^2-x)+b(x^2+1)+c(x-1)$. Comparing coefficients: $a+b=2, \quad -a+c=-1, \quad b-c=-3$. This gives $a=3,\ b=-1,\ c=2$. Hence $[2x^2-x-3]_\beta = \begin{pmatrix}3\\-1\\2\end{pmatrix}$.

Therefore, $Q = \begin{pmatrix}5 & -6 & 3\\ 0 & 4 & -1\\ 3 & -1 & 2\end{pmatrix}$.

(f). $\beta = \{2x^2-x+1, x^2+3x-2, -x^2+2x+1\}$ and $\beta' = \{9x-9, x^2+21x-2, 3x^2+5x+2\}$

Xingjian's Solution: For $9x-9$, write $9x-9 = a(2x^2-x+1)+b(x^2+3x-2)+c(-x^2+2x+1)$. Comparing coefficients: $2a+b-c=0, -a+3b+2c=9, a-2b+c=-9$. This gives $a=-2,\ b=3,\ c=-1$. Hence $[9x-9]_\beta = \begin{pmatrix}-2\\3\\-1\end{pmatrix}$.

For $x^2+21x-2$, write $x^2+21x-2 = a(2x^2-x+1)+b(x^2+3x-2)+c(-x^2+2x+1)$. Comparing coefficients: $2a+b-c=1, -a+3b+2c=21, a-2b+c=-2$. This gives $a=1,\ b=4,\ c=5$. Hence $[x^2+21x-2]_\beta = \begin{pmatrix}1\\4\\5\end{pmatrix}$.

For $3x^2+5x+2$, write $3x^2+5x+2 = a(2x^2-x+1)+b(x^2+3x-2)+c(-x^2+2x+1)$. Comparing coefficients: $2a+b-c=3, -a+3b+2c=5, a-2b+c=2$. This gives $a=2,\ b=1,\ c=2$. Hence $[3x^2+5x+2]_\beta = \begin{pmatrix}2\\1\\2\end{pmatrix}$.

Therefore,

$$Q = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}.$$

Question 5: Let T be the linear operator on $P_1(\mathbb{R})$ defined by $T(p(x)) = p'(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1+x, 1-x\}$. Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

to find $[T]_{\beta'}$.

Xingjian's Solution: Since $\beta = \{1, x\}$, we first compute the matrix of T relative to β. We have $T(1) = 0$, $T(x) = 1$. Thus

$$[T(1)]_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad [T(x)]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence

$$[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Next, let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Since $\beta' = \{1+x,\ 1-x\}$, the columns of Q are $[1+x]_\beta$ and $[1-x]_\beta$. Therefore

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

By the given formula,

$$Q^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

By Theorem 2.23,

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q.$$

Now compute:

$$[T]_\beta Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$[T]_{\beta'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Therefore,

$$\boxed{[T]_{\beta'} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.}$$

Question 7: In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

(a). T is the reflection of $\mathbb{R}^2$ about L.

(b). T is the projection on L along the line perpendicular to L. (See the definition of projection in the exercises of Section 2.1.)

Xingjian's Solution:

Let $u = \begin{pmatrix} 1 \\ m \end{pmatrix}$. Then u is a direction vector for the line $L : y = mx$. Also, $\|u\|^2 = 1+m^2$.

(a). Since T is the reflection about the line L, we use the formula $T(v) = 2\,\mathrm{proj}_L(v) - v$. For $v = \begin{pmatrix} x \\ y \end{pmatrix}$, the projection onto L is

$$\mathrm{proj}_L(v) = \frac{v \cdot u}{u \cdot u} u = \frac{x + my}{1 + m^2} \begin{pmatrix} 1 \\ m \end{pmatrix}.$$

Therefore

$$T(x, y) = 2\frac{x + my}{1 + m^2}(1, m) - (x, y).$$

Now simplify each component:

$$T(x, y) = \left(\frac{2(x + my)}{1 + m^2} - x,\ \frac{2m(x + my)}{1 + m^2} - y \right).$$

Thus

$$T(x, y) = \left(\frac{(1 - m^2)x + 2my}{1 + m^2},\ \frac{2mx + (m^2 - 1)y}{1 + m^2} \right).$$

Hence the reflection about L is

$$\boxed{T(x, y) = \left(\frac{(1 - m^2)x + 2my}{1 + m^2}, \frac{2mx + (m^2 - 1)y}{1 + m^2} \right).}$$

(b). Since T is the projection onto L along the perpendicular line, it is just the orthogonal projection onto L:

$$T(v) = \mathrm{proj}_L(v).$$

Thus

$$T(x, y) = \frac{x + my}{1 + m^2}(1, m).$$

So

$$\boxed{T(x, y) = \left(\frac{x + my}{1 + m^2}, \frac{mx + m^2 y}{1 + m^2} \right).}$$

Equivalently,

$$\boxed{T(x, y) = \left(\frac{x + my}{1 + m^2}, \frac{m(x + my)}{1 + m^2} \right).}$$

Question 10: Prove that if A and B are similar $n \times n$ matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

Xingjian's Solution:

Suppose that A and B are similar $n \times n$ matrices. Then there exists an invertible matrix Q such that $B = Q^{-1}AQ$. Taking trace on both sides, we obtain

$$\mathrm{tr}(B) = \mathrm{tr}(Q^{-1}AQ).$$

Now use the cyclic property of trace: $\mathrm{tr}(XYZ) = \mathrm{tr}(ZXY)$, whenever the products are defined. Hence

$$\mathrm{tr}(Q^{-1}AQ) = \mathrm{tr}(AQQ^{-1}) = \mathrm{tr}(AI) = \mathrm{tr}(A).$$

Therefore,

$$\boxed{\mathrm{tr}(A) = \mathrm{tr}(B).}$$

This proves that similar matrices have the same trace.

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