Question 21: Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, ..., v_n\}$ and $\gamma = \{w_1, w_2, ..., w_m\}$, respectively. By Theorem 2.6 (p. 72), there exist linear transformations $\mathbf{T}_{ij} : \mathbf{V} \to \mathbf{W}$ such that

$$\mathbf{T}_{ij}(v_k) = \begin{cases} w_i, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$$

First prove that $\{\mathbf{T}_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $\mathcal{L}(\mathbf{V}, \mathbf{W})$. Then let $\mathbf{M}^{ij}$ be the $m \times n$ matrix with 1 in the ith row and jth column and 0 elsewhere, and prove that $[\mathbf{T}_{ij}]_\beta^\gamma = \mathbf{M}^{ij}$. Again by Theorem 2.6, there exists a linear transformation $\Phi : \mathcal{L}(\mathbf{V}, \mathbf{W}) \to \mathbf{M}_{m\times n}(F)$ such that $\Phi(\mathbf{T}_{ij}) = M^{ij}$. Prove that Φ is an isomorphism.

Xingjian's Solution:

Proof. **Step 1: Show that $\{T_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a basis for $\mathcal{L}(V, W)$.**

We first show that these transformations $T_{ij}(v_k)$ span $\mathcal{L}(V, W)$.

Let $T \in \mathcal{L}(V, W)$. Since $\gamma = \{w_1, \ldots, w_m\}$ is a basis for W, for each $j = 1, \ldots, n$ we may write $T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m$ for some scalars $a_{ij} \in F$.

Consider the linear combination $S = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}T_{ij}$. Then for each basis vector v_k, $S(v_k) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}T_{ij}(v_k)$. But $T_{ij}(v_k) = 0$ unless $j = k$, so this becomes

$$S(v_k) = \sum_{i=1}^{m} a_{ik}T_{ik}(v_k) = \sum_{i=1}^{m} a_{ik}w_i = T(v_k).$$

Thus S and T agree on every basis vector of V, hence $S = T$. Therefore $\{T_{ij}\}$ spans $\mathcal{L}(V, W)$.

Now we show linear independence. Suppose $\sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}T_{ij} = 0$, where the zero on the right is the zero transformation. Apply both sides to v_k. Then

$$0 = \left(\sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}T_{ij} \right)(v_k) = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}T_{ij}(v_k) = \sum_{i=1}^{m} c_{ik}w_i.$$

Since $\{w_1, \ldots, w_m\}$ is linearly independent, it follows that

$$c_{1k} = c_{2k} = \cdots = c_{mk} = 0.$$

Because k was arbitrary, all coefficients $c_{ij} = 0$. Hence $\{T_{ij}\}$ is linearly independent.

Therefore, $\{T_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a basis for $\mathcal{L}(V, W)$.

Step 2: Show that $[T_{ij}]_\beta^\gamma = M^{ij}$.

By definition, the j-th column of $[T_{ij}]_\beta^\gamma$ is the coordinate vector of $T_{ij}(v_j)$ relative to γ, and the k-th column is the coordinate vector of $T_{ij}(v_k)$.

Now $T_{ij}(v_k) = 0 \quad$ if $k \neq j$, so every column except the j-th is the zero column. Also, $T_{ij}(v_j) = w_i$, whose coordinate vector relative to γ is the standard basis vector $e_i \in F^m$, namely the column with a 1 in row i and zeros elsewhere.

Therefore the matrix $[T_{ij}]_\beta^\gamma$ has all zero columns except the j-th column, which is e_i. Hence it is exactly the matrix M^{ij}, the $m \times n$ matrix having a 1 in the (i, j)-entry and 0 elsewhere:

$$[T_{ij}]_\beta^\gamma = M^{ij}.$$

Step 3: Prove that $\Phi : \mathcal{L}(V,W) \to M_{m\times n}(F)$ defined by $\Phi(T_{ij}) = M^{ij}$ is an isomorphism.

Since $\{T_{ij}\}$ is a basis for $\mathcal{L}(V,W)$, by Theorem 2.6 there exists a unique linear transformation $\Phi : \mathcal{L}(V,W) \to M_{m\times n}(F)$ such that $\Phi(T_{ij}) = M^{ij}$ for all i, j.

To prove Φ is an isomorphism, it is enough to show that it maps a basis of $\mathcal{L}(V,W)$ to a basis of $M_{m\times n}(F)$. But the set $\{M^{ij} : 1 \le i \le m,\ 1 \le j \le n\}$ is the standard basis for $M_{m\times n}(F)$. Therefore Φ sends the basis $\{T_{ij}\}$ of $\mathcal{L}(V,W)$ to the basis $\{M^{ij}\}$ of $M_{m\times n}(F)$.

Hence Φ is one-to-one and onto, so Φ is an isomorphism. □

Question 22: Let $c_0, c_1, ..., c_n$ be distinct scalars from an infinite field F. Define $\mathbf{T} : \mathbf{P}_n(F) \to F^{n+1}$ by $\mathbf{T}(f) = (f(c_0), f(c_1), ..., f(c_n))$. Prove that $\mathbf{T}$ is an isomorphism. Hint: Use the Lagrange polynomials associated with $c_0, c_1, ..., c_n$.

Xingjian's Solution:

Proof. We use the Lagrange polynomials associated with the distinct scalars $c_0, c_1, \ldots, c_n$. For each $j = 0, 1, \ldots, n$, define

$$L_j(x) = \prod_{\substack{0\le k\le n\\ k\ne j}} \frac{x - c_k}{c_j - c_k}.$$

Since the c_k are distinct, each denominator $c_j - c_k$ is nonzero, so $L_j(x)$ is well-defined. Also $L_j \in P_n(F)$, and it satisfies

$$L_j(c_i) = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

One-to-one(Injective): Suppose $T(f) = 0$. Then $f(c_0) = f(c_1) = \cdots = f(c_n) = 0$. Thus f has $n+1$ distinct roots. But $f \in P_n(F)$, so $\deg f \le n$. A nonzero polynomial of degree at most n cannot have more than n distinct roots. Therefore $f = 0$. Hence $\ker T = \{0\}$, so T is injective.

Onto(Surjective): Let $(a_0, a_1, \ldots, a_n) \in F^{n+1}$. Define $f(x) = \sum_{j=0}^{n} a_j L_j(x)$. Since each $L_j \in P_n(F)$, we have $f \in P_n(F)$. Now for each $i = 0, 1, \ldots, n$, $f(c_i) = \sum_{j=0}^{n} a_j L_j(c_i) = a_i$. Therefore

$$\mathbf{T}(f) = (f(c_0), f(c_1), \ldots, f(c_n)) = (a_0, a_1, \ldots, a_n).$$

So $\mathbf{T}$ is surjective.

Since $\mathbf{T}$ is both injective and surjective, it is an isomorphism.

□

Question 23: Let $\mathbf{V}$ denote the vector space defined in Example 5 of Section 1.2, and let $\mathbf{W} = \mathbf{P}(F)$. Define

$$\mathbf{T} : \mathbf{V} \to \mathbf{W} \text{ by } \mathbf{T}(\sigma) = \sum_{i=0}^{n} \sigma(i)x^i,$$

where n is the largest integer such that $\sigma(n) \ne 0$. Prove that $\mathbf{T}$ is an isomorphism.

Xingjian's Solution:

Proof. Remind: Let V be the vector space from Example 5 of Section 1.2, namely the set of all sequences $\sigma = (\sigma(0), \sigma(1), \sigma(2), \dots)$ over F such that only finitely many terms are nonzero. We need to show the following:

Step 1: T is linear.

Let $\sigma, \tau \in V$ and $a \in F$. Then

$$T(\sigma + \tau) = \sum_{i=0}^{\infty} (\sigma(i) + \tau(i))x^i = \sum_{i=0}^{\infty} \sigma(i)x^i + \sum_{i=0}^{\infty} \tau(i)x^i = T(\sigma) + T(\tau),$$

where the sums are finite because both sequences have finite support. Similarly,

$$T(a\sigma) = \sum_{i=0}^{\infty} a\sigma(i)x^i = a \sum_{i=0}^{\infty} \sigma(i)x^i = a\,T(\sigma).$$

Hence T is linear.

Step 2: T is one-to-one transformation (injective).

Suppose $T(\sigma) = 0$, the zero polynomial. Then $\sum_{i=0}^{\infty} \sigma(i)x^i = 0$. A polynomial is zero if and only if all its coefficients are zero. Therefore $\sigma(i) = 0 \quad$ for all $i \geq 0$. Thus σ is the zero sequence, so $\ker T = \{0\}$. Hence T is one-to-one map.

Step 3: T is onto transformation (surjective).

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in P(F)$. Define a sequence $\sigma \in V$ by

$$\sigma(i) = \begin{cases} a_i, & 0 \leq i \leq n, \\ 0, & i > n. \end{cases}$$

Then σ has only finitely many nonzero terms, so $\sigma \in V$, and

$$T(\sigma) = \sum_{i=0}^{n} a_i x^i = p(x).$$

Therefore every polynomial in $P(F)$ lies in the image of T, so T is surjective.

Since T is linear, injective, and surjective, it is an isomorphism. □

Question 24: Let $T : V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping

$$\overline{T} : V/N(T) \to Z \qquad \text{by} \qquad \overline{T}(v + N(T)) = T(v)$$

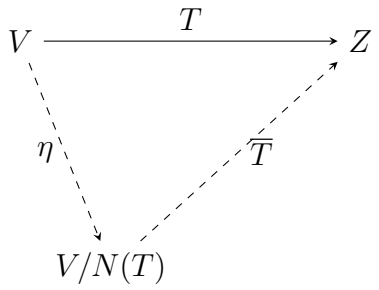
for any coset $v + N(T)$ in $V/N(T)$.

(a). Prove that $\overline{T}$ is well-defined; that is, prove that if $v + N(T) = v' + N(T)$, then $\overline{T}(v) = \overline{T}(v')$.

(b). Prove that $\overline{T}$ is linear.

(c). Prove that $\overline{T}$ is an isomorphism.

(d). Prove that the diagram shown in Figure 2.3 commutes; that is, prove that $T = \overline{T}\eta$.



Xingjian's Solution:

Proof. **(a) $\overline{T}$ is well-defined:** To show $\overline{T}$ is well-defined, we must prove that if $v+N(T) = v'+N(T)$, then $\overline{T}(v+N(T)) = \overline{T}(v'+N(T))$.

Now $v+N(T) = v'+N(T)$ means that $v-v' \in N(T)$. By definition of the null space, $T(v-v') = 0$. Since T is linear, $T(v)-T(v') = 0$, so $T(v) = T(v')$. Therefore,

$$\overline{T}(v+N(T)) = T(v) = T(v') = \overline{T}(v'+N(T)).$$

Hence $\overline{T}$ is well-defined.

(b) $\overline{T}$ is linear: Let $u+N(T), v+N(T) \in V/N(T)$, and let $a \in F$. Then

$$\overline{T}\big((u+N(T))+(v+N(T))\big) = \overline{T}\big((u+v)+N(T)\big) = T(u+v).$$

Since T is linear,

$$T(u+v) = T(u)+T(v) = \overline{T}(u+N(T)) + \overline{T}(v+N(T)).$$

Thus

$$\overline{T}\big((u+N(T))+(v+N(T))\big) = \overline{T}(u+N(T)) + \overline{T}(v+N(T)).$$

Also,

$$\overline{T}\big(a(v+N(T))\big) = \overline{T}(av+N(T)) = T(av) = aT(v) = a\,\overline{T}(v+N(T)).$$

Therefore $\overline{T}$ is linear.

(c) $\overline{T}$ is an isomorphism: Since T is onto Z, for every $z \in Z$ there exists $v \in V$ such that $T(v) = z$. Then $\overline{T}(v+N(T)) = T(v) = z$, so $\overline{T}$ is onto.

Now we show $\overline{T}$ is one-to-one. Suppose $\overline{T}(v+N(T)) = 0$. Then $T(v) = 0$, so $v \in N(T)$. Hence $v+N(T) = N(T)$, which is the zero element of the quotient space $V/N(T)$. Therefore the kernel of $\overline{T}$ is trivial: $N(\overline{T}) = \{N(T)\}$. So $\overline{T}$ is injective.

Since $\overline{T}$ is linear, one-to-one, and onto, it is an isomorphism. □

(d) The diagram commutes; that is, $T = \overline{T}\eta$: Let $\eta : V \to V/N(T)$ be the natural quotient map defined by $\eta(v) = v+N(T)$. Then for every $v \in V$, $(\overline{T}\eta)(v) = \overline{T}(\eta(v)) = \overline{T}(v+N(T)) = T(v)$. Thus $(\overline{T}\eta)(v) = T(v)$ for all $v \in V$, and hence $T = \overline{T}\eta$.

Question 25: Let V be a nonzero vector space over a field F, and suppose that S is a basis for V. (By the corollary to Theorem 1.13 (p. 60) in Section 1.7, every vector space has a basis). Let $C(S,F)$ denote the vector space of all functions $f \in \mathcal{F}(S,F)$ such that $f(s) = 0$ for all but a finite number of vectors in S. (See Exercise 14 of Section 1.3.) Let

$$\Psi : C(S,F) \to V$$

be the function defined by

$$\Psi(f) = \sum_{\substack{s \in S \\ f(s) \neq 0}} f(s)s.$$

Prove that Ψ is an isomorphism. Thus every nonzero vector space can be viewed as a space of functions.

Xingjian's Solution:

Proof. Let S be a basis for the nonzero vector space V, and let

$$C(S,F) = \{f \in \mathcal{F}(S,F) : f(s) = 0 \text{ for all but finitely many } s \in S\}.$$

By definition, since $f(s) = 0$ for all but finitely many $s \in S$, the above sum is finite, so $\Psi(f)$ is well-defined. We prove that Ψ is an isomorphism.

Step 1: Ψ is linear.

Let $f, g \in C(S,F)$ and $a \in F$. Since both f and g have finite support, so do $f+g$ and af. Then

$$\Psi(f+g) = \sum_{s \in S}(f(s)+g(s))s = \sum_{s \in S} f(s)s + \sum_{s \in S} g(s)s = \Psi(f) + \Psi(g),$$

where all sums are finite.

Similarly,

$$\Psi(af) = \sum_{s \in S} af(s)s = a\sum_{s \in S} f(s)s = a\,\Psi(f).$$

Hence Ψ is linear.

Step 2: Ψ is onto.

Let $v \in V$. Since S is a basis for V, v can be written uniquely as a finite linear combination of elements of S:

$$v = a_1 s_1 + \cdots + a_n s_n,$$

where $s_1, \ldots, s_n \in S$ and $a_1, \ldots, a_n \in F$.

Define $f : S \to F$ by

$$f(s) = \begin{cases} a_i, & \text{if } s = s_i \text{ for some } i = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in C(S, F)$, because $f(s) \neq 0$ for only finitely many $s \in S$. Moreover,

$$\Psi(f) = \sum_{i=1}^{n} a_i s_i = v.$$

Thus Ψ is onto.

Step 3: Ψ is one-to-one.

Suppose $\Psi(f) = 0$. Then

$$\sum_{\substack{s \in S \\ f(s) \neq 0}} f(s)s = 0.$$

This is a finite linear combination of vectors in the basis S. Since a basis is linearly independent, every coefficient in this sum must be zero. Therefore

$$f(s) = 0 \qquad \text{for every } s \in S.$$

So f is the zero function. Hence $\ker \Psi = \{0\}$, and Ψ is injective.

Since Ψ is linear, onto, and one-to-one, it follows that Ψ is an isomorphism.

Therefore every nonzero vector space is isomorphic to a vector space of functions with finite support. □

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