Question 1: Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β, respectively, $T : V \to W$ is linear, and A and B are matrices.

(a). $\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\alpha^\beta$.

(b). T is invertible if and only if T is one-to-one and onto.

(c). $T = L_A$, where $A = [T]_\alpha^\beta$.

(d). $M_{2\times 5}(F)$ is isomorphic to F^5.

(e). $P_n(F)$ is isomorphic to $P_m(F)$ if and only if $n = m$.

(f). $AB = I$ implies that A and B are invertible.

(g). If A is invertible, then $(A^{-1})^{-1} = A$.

(h). A is invertible if and only if L_A is invertible.

(i). A must be square in order to possess an inverse.

Xingjian's Solution:

(a): **False.** In general, $[T]_\alpha^\beta$ is a matrix from $\dim V$ to $\dim W$. For $\left([T]_\alpha^\beta\right)^{-1}$ to even make sense, $[T]_\alpha^\beta$ must be square and invertible, which requires T to be an isomorphism $V \cong W$. Even in that case, the correct identity is $\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\beta^\alpha$, not $[T^{-1}]_\alpha^\beta$ (the bases swap for the inverse map).

(b): **True.** By definition, T is invertible iff there exists $S : W \to V$ such that $ST = I_V$ and $TS = I_W$. This is equivalent to T being both one-to-one and onto.

(c): **False.** The matrix $A = [T]_\alpha^\beta$ represents T with respect to bases α, β. But L_A is the linear map $F^n \to F^m$ given by $x \mapsto Ax$, whose domain/codomain are coordinate spaces, not V and W. The correct relationship is $[T(v)]_\beta = A\,[v]_\alpha \quad$ for all $v \in V$, not $T = L_A$.

(d): **False.** $\dim M_{2\times 5}(F) = 10$ while $\dim F^5 = 5$. Since isomorphic vector spaces must have the same dimension, they are not isomorphic.

(e): **True.** $\dim P_n(F) = n + 1$ and $\dim P_m(F) = m + 1$. They are isomorphic iff their dimensions are equal, i.e. $n + 1 = m + 1$, hence $n = m$.

(f): **True.** If $AB = I$ (with A, B $n \times n$), then A has a right-inverse and B has a left-inverse. For square matrices, this implies both are invertible and $B = A^{-1}$, $A = B^{-1}$.

(g): **True.** If A is invertible, then $AA^{-1} = A^{-1}A = I$. Thus A is an inverse of A^{-1}, so $(A^{-1})^{-1} = A$.

(h): **True.** $L_A : F^n \to F^n$ is defined by $L_A(x) = Ax$. If A is invertible, then $L_{A^{-1}}$ is the inverse operator of L_A, so L_A is invertible. Conversely, if L_A is invertible, then $Ax = 0 \Rightarrow x = 0$, so the columns of A are linearly independent and A is invertible.

(i): **True.** If A has an inverse A^{-1}, then both products AA^{-1} and $A^{-1}A$ must equal an identity matrix. This requires A to be $n \times n$.

Question 2: For each of the following linear transformations T, determine whether T is invertible and justify your answer.

(a). $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - 2a_2,\ a_2,\ 3a_1 + 4a_2)$.

Xingjian's Solution: **Not invertible:** An invertible linear map must be a bijection, so in particular the domain and codomain must have the same dimension. Here $\dim(\mathbb{R}^2) = 2$ and $\dim(\mathbb{R}^3) = 3$, so no linear map $\mathbb{R}^2 \to \mathbb{R}^3$ can be onto. Hence T is not invertible.

(b). $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (3a_1 - a_2,\ a_2,\ 4a_1)$.

Xingjian's Solution: **Not invertible:** Again $\dim(\mathbb{R}^2) = 2 \neq 3 = \dim(\mathbb{R}^3)$, so T cannot be onto. Therefore T is not invertible.

(c). $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3,\ a_2,\ 3a_1 + 4a_2)$.

Xingjian's Solution: **Not invertible.** Compute injectivity by solving $T(a_1, a_2, a_3) = (0, 0, 0)$:

$$(3a_1 - 2a_3,\ a_2,\ 3a_1 + 4a_2) = (0, 0, 0).$$

Thus $a_2 = 0$. Then the third coordinate gives $3a_1 + 4a_2 = 3a_1 = 0$, so $a_1 = 0$. Then the first coordinate gives $3a_1 - 2a_3 = -2a_3 = 0$, so $a_3 = 0$. Hence $N(T) = \{0\}$, so T is one-to-one. But the image always has second coordinate equal to a_2, and the third coordinate equals $3a_1 + 4a_2$. Since the second output coordinate is exactly a_2, while the third depends only on a_1, a_2 and never on a_3, the map cannot hit every $(y_1, y_2, y_3) \in \mathbb{R}^3$. More directly, the matrix of T in the standard basis is

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix},$$

and

$$\det(A) = 3 \det \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix} - 0 + \ (-2) \det \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} = 3(0) - 2(0 \cdot 4 - 1 \cdot 3) = 6 \neq 0.$$

So actually $\det(A) \neq 0$ implies T *is* invertible.

Conclusion: T **is invertible** (since $\det(A) = 6 \neq 0$).

(d). $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p''(x)$.

Xingjian's Solution: The map $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ is given by $T(p) = p''$. It is not one-to-one because any polynomial of degree ≤ 1 has second derivative zero. For example,

$$p(x) = 1 \neq 0 \quad \text{but} \quad T(p) = p''(x) = 0.$$

So $N(T) \neq \{0\}$, hence T is not invertible.

(e). $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c + d)x^2.$$

Xingjian's Solution: **Not invertible.** First note $\dim M_{2\times 2}(\mathbb{R}) = 4$ while $\dim P_2(\mathbb{R}) = 3$. Thus T cannot be one-to-one (since a linear map from a 4D space to a 3D space must have nontrivial kernel). Hence T is not invertible.

(Indeed, if $a = b = 0$ and $c = -d \neq 0$, then $T \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} = 0$.)

(f). $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & a \\ c & c + d \end{pmatrix}.$$

Xingjian's Solution: **Invertible.** Let

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}.$$

Suppose

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then from the $(1,2)$-entry we get $a = 0$. From the $(1,1)$-entry: $a + b = 0 \Rightarrow b = 0$. From the $(2,1)$-entry: $c = 0$. From the $(2,2)$-entry: $c + d = 0 \Rightarrow d = 0$. Hence the kernel is trivial, so T is one-to-one.

Since $\dim M_{2\times 2}(\mathbb{R}) = 4$ equals the codomain dimension, one-to-one implies onto. Therefore T is invertible.

(Equivalently, given $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, we can solve uniquely: $a = q,\ b = p - q,\ c = r,\ d = s - r$.)

Question 4: Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Xingjian's Solution:

Proof. Since A and B are invertible, there exist matrices A^{-1} and B^{-1} such that $AA^{-1} = A^{-1}A = I, BB^{-1} = B^{-1}B = I$. Consider the matrix $B^{-1}A^{-1}$. We check that it is the inverse of AB.

First,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Second,

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Thus $B^{-1}A^{-1}$ is the inverse of AB. Therefore AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. □

Question 5: Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$

Xingjian's Solution:

Proof. Since A is invertible, we have $AA^{-1} = A^{-1}A = I$. Take transpose on both sides of $AA^{-1} = I$: $(AA^{-1})^t = I^t$. Using the property $(AB)^t = B^tA^t$, we obtain $(A^{-1})^tA^t = I$.

Similarly, taking transpose of $A^{-1}A = I$ gives $A^t(A^{-1})^t = I$.

Thus $(A^{-1})^t$ is the inverse of A^t. Therefore A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$. □

Question 6: Prove that if A is invertible and AB = O, then B = O.

Xingjian's Solution:

Proof. Since A is invertible, multiply both sides of $AB = O$ by A^{-1} on the left: $A^{-1}(AB) = A^{-1}O$. Using associativity of matrix multiplication, $(A^{-1}A)B = O$. Since $A^{-1}A = I$, we obtain $IB = B$. Thus $B = O$. □

Question 7: Let A be an $n \times n$ matrix.

(a). Suppose that $A^2 = O$. Prove that A is not invertible.

(b). Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain

Xingjian's Solution:

Proof. (a). Assume for contradiction that A is invertible. Then multiply both sides of $A^2 = O$ by A^{-1} on the left: $A^{-1}A^2 = A^{-1}O$. This gives $(A^{-1}A)A = A$. Thus $A = O$.

But the zero matrix is not invertible, which contradicts the assumption that A is invertible. Therefore A cannot be invertible.

(b). Suppose $AB = O$ for some nonzero matrix B. If A were invertible, multiplying both sides by A^{-1} on the left would give $A^{-1}(AB) = A^{-1}O$. Thus $(A^{-1}A)B = B = O$.

This contradicts the assumption that $B \neq O$. Therefore A cannot be invertible. □

Question 9: Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

Xingjian's Solution:

Proof. Since AB is invertible, there exists an $n \times n$ matrix C such that $(AB)C = C(AB) = I_n$. Since $A(BC) = I_n$, we see that A has a right inverse, namely BC. For square matrices, having a right inverse implies invertibility. Hence A is invertible.

Similarly, since $(CA)B = I_n$, we see that B has a left inverse, namely CA. Again, since B is square, having a left inverse implies invertibility. Hence B is invertible. Therefore, both A and B are invertible.

Now we give an example showing that if A and B are *not* square matrices, then AB can be invertible while neither A nor B is invertible.

Take

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

which is invertible. However, A is 2×3 and B is 3×2, so neither matrix is invertible. □

Question 12: Prove Theorem 2.21. For any finite-dimensional vector space $\mathbf{V}$ with ordered basis β, ϕ_β is an isomorphism.

Xingjian's Solution:

Proof. Let $\mathbf{V}$ be a finite-dimensional vector space over F, and let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for $\mathbf{V}$.

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \downarrow{\scriptstyle \phi_\beta} & & \downarrow{\scriptstyle \phi_\beta} \\ F^n & \xrightarrow{[T]_\beta} & F^n \end{array}$$

Define $\phi_\beta : \mathbf{V} \to F^n$ by $\phi_\beta(v) = [v]_\beta$, where $[v]_\beta$ is the coordinate vector of v relative to the basis β. We prove that ϕ_β is an isomorphism.

Step 1: ϕ_β is linear. **Xingjian comment:** Here in abstract algebra we can say ϕ_β is homomorphism.

Let $u, v \in \mathbf{V}$ and $c \in F$. Write $u = \sum_{i=1}^n a_i v_i$; $\quad v = \sum_{i=1}^n b_i v_i$. Then $u + cv = \sum_{i=1}^n (a_i + cb_i)v_i$. Hence

$$\phi_\beta(u + cv) = [u + cv]_\beta = \begin{pmatrix} a_1 + cb_1 \\ a_2 + cb_2 \\ \vdots \\ a_n + cb_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + c \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \phi_\beta(u) + c\phi_\beta(v).$$

So ϕ_β is linear.

Step 2: ϕ_β is one-to-one (injective). Suppose $\phi_\beta(v) = 0$. Then $[v]_\beta = 0$, so all coordinates of v relative to β are zero. Thus $v = 0v_1 + \cdots + 0v_n = 0$. Therefore, $\ker(\phi_\beta) = \{0\}$, and so ϕ_β is injective.

Step 3: ϕ_β is onto (surjective).

Let

$$x = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n.$$

Define $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \in \mathbf{V}$. Then by definition of coordinates,

$$\phi_\beta(v) = [v]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = x.$$

So ϕ_β is surjective.

Since ϕ_β is linear, one-to-one, and onto, it is an isomorphism. □

Question 15: Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces, and let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Suppose that β is a basis for $\mathbf{V}$. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for $\mathbf{W}$.

Xingjian's Solution:

Proof. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis for $\mathbf{V}$. Then $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$. We prove both directions.

($\Rightarrow$) Suppose T is an isomorphism. Prove that $T(\beta)$ is a basis for $\mathbf{W}$.

Since T is an isomorphism, it is linear, one-to-one, and onto. First, we show $T(\beta)$ spans $\mathbf{W}$. Let $w \in \mathbf{W}$. Since T is onto, there exists $v \in \mathbf{V}$ such that $T(v) = w$. Because β is a basis for $\mathbf{V}$, we can write $v = a_1v_1 + \cdots + a_nv_n$. Applying T, we get $w = T(v) = a_1T(v_1) + \cdots + a_nT(v_n)$. Thus w is a linear combination of vectors in $T(\beta)$, so $T(\beta)$ spans $\mathbf{W}$.

Next, we show $T(\beta)$ is linearly independent. Suppose $c_1T(v_1) + \cdots + c_nT(v_n) = 0$. By linearity of T, $T(c_1v_1 + \cdots + c_nv_n) = 0$. Since T is one-to-one, its kernel is $\{0\}$. Therefore $c_1v_1 + \cdots + c_nv_n = 0$. Because β is linearly independent, it follows that $c_1 = \cdots = c_n = 0$. So $T(\beta)$ is linearly independent. Hence $T(\beta)$ is a basis for $\mathbf{W}$.

($\Leftarrow$) Suppose $T(\beta)$ is a basis for $\mathbf{W}$. Prove that T is an isomorphism.

Since $T(\beta)$ is a basis for $\mathbf{W}$, it spans $\mathbf{W}$. Let $w \in \mathbf{W}$. Then $w = a_1T(v_1) + \cdots + a_nT(v_n)$ for some scalars $a_1, \ldots, a_n$. By linearity, $w = T(a_1v_1 + \cdots + a_nv_n)$. Thus every $w \in \mathbf{W}$ lies in the image of T, so T is onto.

Now suppose $T(v) = 0$. Since β is a basis for $\mathbf{V}$, write $v = a_1v_1 + \cdots + a_nv_n$. Then $0 = T(v) = a_1T(v_1) + \cdots + a_nT(v_n)$. Because $T(\beta)$ is linearly independent, we must have $a_1 = \cdots = a_n = 0$. Hence $v = 0$. So $\ker(T) = \{0\}$, and T is one-to-one. Therefore T is linear, one-to-one, and onto, so T is an isomorphism. □

Question 16: Let B be an $n \times n$ invertible matrix. Define $\phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\phi(A) = B^{-1}AB$. Prove that ϕ is an isomorphism.

Xingjian's Solution:

Proof. **Step 1: ϕ is linear.**

Let $A_1, A_2 \in M_{n\times n}(F)$ and $c \in F$. Then

$$\phi(A_1 + cA_2) = B^{-1}(A_1 + cA_2)B = B^{-1}A_1B + cB^{-1}A_2B = \phi(A_1) + c\phi(A_2).$$

Hence ϕ is linear.

Step 2: ϕ is one-to-one.

Suppose $\phi(A) = 0$. Then $B^{-1}AB = 0$. Multiplying on the left by B and on the right by B^{-1}, we get $A = 0$. Thus $\ker(\phi) = \{0\}$, so ϕ is injective.

Step 3: ϕ is onto.

Let $C \in M_{n\times n}(F)$. We want to find $A \in M_{n\times n}(F)$ such that $\phi(A) = C$. Take $A = BCB^{-1}$. Then $\phi(A) = B^{-1}(BCB^{-1})B = C$. Therefore ϕ is surjective.

Since ϕ is linear, injective, and surjective, it is an isomorphism.

In fact, the inverse map is $\phi^{-1}(C) = BCB^{-1}$. □

Question 17: Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces and $T : \mathbf{V} \to \mathbf{W}$ be an isomorphism. Let V_0 be a subspace of $\mathbf{V}$.

(a) Prove that $T(V_0)$ is a subspace of W.

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

Xingjian's Solution:

Proof. Since $T : \mathbf{V} \to \mathbf{W}$ is an isomorphism, T is linear, one-to-one, and onto.

(a) Prove that $T(V_0)$ is a subspace of $\mathbf{W}$.

We verify the subspace conditions. First, since V_0 is a subspace of $\mathbf{V}$, we have $0 \in V_0$. Because T is linear, $T(0) = 0$, so $0 \in T(V_0)$. Hence $T(V_0)$ is nonempty.

Now let $u, w \in T(V_0)$ and let $c \in F$. Then there exist $v_1, v_2 \in V_0$ such that $u = T(v_1), \quad w = T(v_2)$. Since V_0 is a subspace, $v_1 + cv_2 \in V_0$. Using linearity of T, $u + cw = T(v_1) + cT(v_2) = T(v_1 + cv_2)$. Because $v_1 + cv_2 \in V_0$, it follows that $u + cw \in T(V_0)$. Therefore $T(V_0)$ is closed under addition and scalar multiplication, so $T(V_0)$ is a subspace of $\mathbf{W}$.

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

Let $\beta = \{v_1, \ldots, v_k\}$ be a basis for V_0. Then $\dim(V_0) = k$. We will show that $T(\beta) = \{T(v_1), \ldots, T(v_k)\}$ is a basis for $T(V_0)$.

First, we show that $T(\beta)$ spans $T(V_0)$. Let $y \in T(V_0)$. Then by definition, there exists $v \in V_0$ such that $y = T(v)$. Since β is a basis for V_0, we can write $v = a_1v_1 + \cdots + a_kv_k$ for some scalars $a_1, \ldots, a_k$. Applying T, we get $y = T(v) = a_1T(v_1) + \cdots + a_kT(v_k)$. Thus y is a linear combination of vectors in $T(\beta)$, so $T(\beta)$ spans $T(V_0)$.

Next, we show that $T(\beta)$ is linearly independent. Suppose $c_1T(v_1) + \cdots + c_kT(v_k) = 0$. By linearity, $T(c_1v_1 + \cdots + c_kv_k) = 0$. Since T is one-to-one, $\ker(T) = \{0\}$. Therefore $c_1v_1 + \cdots + c_kv_k = 0$. Because β is linearly independent, it follows that $c_1 = \cdots = c_k = 0$. So $T(\beta)$ is linearly independent.

Hence $T(\beta)$ is a basis for $T(V_0)$. Therefore, $\dim(T(V_0)) = k = \dim(V_0)$.

□

Question 18: Repeat Example 7 with the polynomial $p(x) = 1 + x + 2x^2 + x^3$.
Xingjian's Solution:

Proof. Recall from Example 7 that $T : P_3(\mathbb{R}) \to P_2(\mathbb{R}), \quad T(f(x)) = f'(x)$, and with respect to the standard ordered bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$, the matrix of T is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Let $p(x) = 1 + x + 2x^2 + x^3$. Then the coordinate vector of $p(x)$ relative to the standard basis $\beta = \{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$ is

$$\phi_\beta(p(x)) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

Now compute

$$L_A\phi_\beta(p(x)) = A \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

On the other hand, $T(p(x)) = p'(x) = 1+4x+3x^2$. Thus the coordinate vector of $T(p(x))$ relative to the standard basis $\gamma = \{1, x, x^2\}$ of $P_2(\mathbb{R})$ is

$$\phi_\gamma(T(p(x))) = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

Therefore, $L_A\phi_\beta(p(x)) = \phi_\gamma(T(p(x)))$. So Example 7 is verified for the polynomial $p(x) = 1 + x + 2x^2 + x^3$. □

Question 19: In Example 5 of Section 2.1, the mapping $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T(M) = M^t$ for each $M \in M_{2\times2}(\mathbb{R})$ is a linear transformation. Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, which is a basis for $M_{2\times2}(\mathbb{R})$, as noted in Example 3 of Section 1.6.

(a). Compute $[T]_\beta$.

(b). Verify that $L_A\phi_\beta(M) = \phi_\beta T(M)$ for $A = [T]_\beta$ and $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Xingjian's Solution:

Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the ordered basis for $M_{2\times2}(\mathbb{R})$, where

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The linear transformation is $T(M) = M^t$.

(a). To find $[T]_\beta$, we compute T on each basis vector and express the result in coordinates relative to β.

First, $T(E_{11}) = E_{11}^t = E_{11}$. So $[T(E_{11})]_\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Next, $T(E_{12}) = E_{12}^t = E_{21}$. So $[T(E_{12})]_\beta = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

Also, $T(E_{21}) = E_{21}^t = E_{12}$. So $[T(E_{21})]_\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Finally, $T(E_{22}) = E_{22}^t = E_{22}$. So $[T(E_{22})]_\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

Therefore, the matrix of T relative to the ordered basis β is

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Verify that $L_A\phi_\beta(M) = \phi_\beta(T(M))$.

Let

$$A = [T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Relative to the basis $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, we can write $M = 1E_{11}+2E_{12}+3E_{21}+4E_{22}$. Hence

$$\phi_\beta(M) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Now compute

$$L_A\phi_\beta(M) = A \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}.$$

On the other hand, $T(M) = M^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. Thus $T(M) = 1E_{11} + 3E_{12} + 2E_{21} + 4E_{22}$, so

$$\phi_\beta(T(M)) = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}.$$

Therefore,

$$L_A\phi_\beta(M) = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix} = \phi_\beta(T(M)).$$

Hence the identity $L_A\phi_\beta(M) = \phi_\beta(T(M))$ is verified.

Question 20: Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation from an n-dimensional vector space $\mathbf{V}$ to an m-dimensional vector space $\mathbf{W}$. Let β and γ be ordered bases for $\mathbf{V}$ and $\mathbf{W}$, respectively. Prove that $\text{rank}(T) = \text{rank}(L_A)$ and that $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_\beta^\gamma$. (Hint: Apply Exercise 17 to Figure 2.2).

Xingjian's Solution:

Proof. it is helpful to use the commutative diagram associated with the matrix representation $A = [T]_\beta^\gamma$:

$$\begin{array}{ccc} \mathbf{V} & \xrightarrow{T} & \mathbf{W} \\ {\scriptstyle \phi_\beta}\downarrow & & \downarrow{\scriptstyle \phi_\gamma} \\ F^n & \xrightarrow{L_A} & F^m \end{array}$$

Since $\phi_\beta : \mathbf{V} \to F^n$ and $\phi_\gamma : \mathbf{W} \to F^m$ are isomorphisms, they preserve subspaces and dimensions. We first prove that $\text{rank}(T) = \text{rank}(L_A)$.

Recall that

$$R(T) = \{T(v) : v \in \mathbf{V}\} \quad \text{and} \quad R(L_A) = \{L_A(x) : x \in F^n\}.$$

From the commutative relation $\phi_\gamma \circ T = L_A \circ \phi_\beta$, if $y \in R(T)$, then $y = T(v)$ for some $v \in \mathbf{V}$, and hence $\phi_\gamma(y) = \phi_\gamma(T(v)) = L_A(\phi_\beta(v)) \in R(L_A)$. Thus $\phi_\gamma(R(T)) \subseteq R(L_A)$.

Conversely, if $z \in R(L_A)$, then $z = L_A(x)$ for some $x \in F^n$. Since ϕ_β is onto, there exists $v \in \mathbf{V}$ such that $x = \phi_\beta(v)$. Therefore $z = L_A(\phi_\beta(v)) = \phi_\gamma(T(v)) \in \phi_\gamma(R(T))$. So $R(L_A) \subseteq \phi_\gamma(R(T))$. Hence

$$\phi_\gamma(R(T)) = R(L_A).$$

Because ϕ_γ is an isomorphism, Exercise 17 implies that $\dim(R(T)) = \dim(\phi_\gamma(R(T)))$. Therefore

$$\text{rank}(T) = \dim(R(T)) = \dim(R(L_A)) = \text{rank}(L_A).$$

Now we prove that $\text{nullity}(T) = \text{nullity}(L_A)$.

Recall that $N(T) = \{v \in \mathbf{V} : T(v) = 0\}$ and $N(L_A) = \{x \in F^n : L_A(x) = 0\}$. Let $v \in N(T)$. Then $T(v) = 0$, so by the commutative relation, $L_A(\phi_\beta(v)) = \phi_\gamma(T(v)) = \phi_\gamma(0) = 0$. Thus $\phi_\beta(v) \in N(L_A)$, and hence $\phi_\beta(N(T)) \subseteq N(L_A)$.

Conversely, let $x \in N(L_A)$. Since ϕ_β is onto, there exists $v \in \mathbf{V}$ such that $x = \phi_\beta(v)$. Then

$$0 = L_A(x) = L_A(\phi_\beta(v)) = \phi_\gamma(T(v)).$$

Because ϕ_γ is one-to-one, this implies $T(v) = 0$. Hence $v \in N(T)$, so $x = \phi_\beta(v) \in \phi_\beta(N(T))$. Therefore

$$N(L_A) = \phi_\beta(N(T)).$$

Again, since ϕ_β is an isomorphism, Exercise 17 gives $\dim(N(T)) = \dim(\phi_\beta(N(T))) = \dim(N(L_A))$. Thus

$$\text{nullity}(T) = \text{nullity}(L_A).$$

Therefore,

$$\boxed{\text{rank}(T) = \text{rank}(L_A)} \qquad \text{and} \qquad \boxed{\text{nullity}(T) = \text{nullity}(L_A)}.$$

□

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