Question 1: Label the following statements as **true** or **false**. In each part, let V, W, and Z be vector spaces with ordered (finite) bases α, β, and γ, respectively. Let $T : V \to W$ and $U : W \to Z$ be linear transformations, and let A and B denote matrices.

(a). $[UT]_\alpha^\gamma = [T]_\beta^\gamma [U]_\alpha^\beta$.

(b). $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ for all $v \in V$.

(c). $[U(w)]_\beta = [U]_\alpha^\beta [w]_\beta$ for all $w \in W$.

(d). $[I_V]_\alpha = I$.

(e). $[T^2]_\alpha^\beta = ([T]_\alpha^\beta)^2$.

(f). $A^2 = I$ implies that $A = I$ or $A = -I$.

(g). $T = L_A$ for some matrix A.

(h). $A^2 = O$ implies that $A = O$, where O denotes the zero matrix.

(i). $L_{A+B} = L_A + L_B$.

(j). If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$.

Xingjian's Solution:

(a). **False.** The correct composition rule is $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.

(b). **True.** By definition of the matrix of T with respect to bases α and β, $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ for all $v \in V$.

(c). **False.** For $U : W \to Z$, the correct formula is $[U(w)]_\gamma = [U]_\beta^\gamma [w]_\beta$. The indices in the statement do not match the spaces.

(d). **True.** The identity map I_V sends each basis vector to itself, so its matrix in basis α is the identity matrix: $[I_V]_\alpha = I$.

(e). **False in general.** As stated, $T : V \to W$, so T^2 is not even defined unless $V = W$. If $T : V \to V$ and α is a basis of V, then $[T^2]_\alpha = \big([T]_\alpha\big)^2$ is the correct statement.

(f). **False.** For example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfies $A^2 = I$ but $A \neq I$ and $A \neq -I$.

(g). **False in the abstract setting.** L_A denotes left-multiplication by a matrix A on a coordinate space $\mathbb{F}^n$. For general abstract vector spaces V, W, a linear map $T : V \to W$ corresponds to some matrix only after choosing bases and identifying $V \cong \mathbb{F}^n$, $W \cong \mathbb{F}^m$; but T is not literally equal to L_A on V.

(h). **False.** Take

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $A \neq O$ but

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O.$$

(i). **True.** For every vector x of compatible size,

$$L_{A+B}(x) = (A + B)x = Ax + Bx = L_A(x) + L_B(x),$$

so $L_{A+B} = L_A + L_B$.

(j). **True.** The condition $A_{ij} = \delta_{ij}$ exactly says that A has 1's on the diagonal and 0's off the diagonal, hence $A = I$.

Question 2:

(a). Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

Compute

$$A(2B + 3C), \qquad (AB)D, \qquad A(BD).$$

Xingjian's Solution:

First compute

$$2B = \begin{pmatrix} 2 & 0 & -6 \\ 8 & 2 & 4 \end{pmatrix}, \qquad 3C = \begin{pmatrix} 3 & 3 & 12 \\ -3 & -6 & 0 \end{pmatrix}.$$

Thus

$$2B + 3C = \begin{pmatrix} 2 & 0 & -6 \\ 8 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 3 & 12 \\ -3 & -6 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 6 \\ 5 & -4 & 4 \end{pmatrix}.$$

Now

$$A(2B + 3C) = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 6 \\ 5 & -4 & 4 \end{pmatrix} = \begin{pmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{pmatrix}.$$

Next compute

$$AB = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 3 & 3 \\ -2 & -1 & -8 \end{pmatrix}.$$

Then

$$(AB)D = \begin{pmatrix} 13 & 3 & 3 \\ -2 & -1 & -8 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 29 \\ -26 \end{pmatrix}.$$

Compute BD:

$$BD = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ 12 \end{pmatrix}.$$

Finally,

$$A(BD) = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -7 \\ 12 \end{pmatrix} = \begin{pmatrix} 29 \\ -26 \end{pmatrix}.$$

So the results are

$$A(2B + 3C) = \begin{pmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{pmatrix}, \qquad (AB)D = A(BD) = \begin{pmatrix} 29 \\ -26 \end{pmatrix}.$$

(b).

$$A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}, \quad C = (4\ 0\ 3).$$

Compute

$$A^t, \quad A^tB, \quad BC^t, \quad CB, \quad CA.$$

Xingjian's Solution:
First, the transpose:

$$A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}.$$

Then

$$A^tB = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{pmatrix}.$$

Next,

$$C^t = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}, \quad BC^t = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}.$$

Now

$$CB = (4\ 0\ 3) \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 27 & 7 & 9 \end{pmatrix}.$$

Finally,

$$CA = (4\ 0\ 3) \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 20 & 26 \end{pmatrix}.$$

So we have

$$A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}, \quad A^tB = \begin{pmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{pmatrix},$$

$$BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}, \quad CB = \begin{pmatrix} 27 & 7 & 9 \end{pmatrix}, \quad CA = \begin{pmatrix} 20 & 26 \end{pmatrix}.$$

Question 3: Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations defined by

$$T(f(x)) = f'(x)g(x) + 2f(x)$$

and

$$U(a + bx + cx^2) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

1. Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

2. Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from part (a) and Theorem 2.14 to verify your result.

Xingjian's Solution:

Let the standard ordered basis of $P_2(\mathbb{R})$ be $\beta = \{1, x, x^2\}$ and the standard ordered basis of $\mathbb{R}^3$ be $\gamma = \{e_1, e_2, e_3\}$.

For 1: Evaluate U on the basis elements of β with $U(1) = (1, 0, 1)$, $U(x) = (1, 0, -1)$,$U(x^2) = (0, 1, 0)$

Thus the coordinate vectors are

$$[U(1)]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad [U(x)]_\gamma = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad [U(x^2)]_\gamma = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Evaluate T on the basis elements of β with $T(1) = 0 \cdot (3 + x) + 2(1) = 2$

$$[2]_\beta = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

$T(x) = 1(3 + x) + 2x = 3 + 3x$

$$[3 + 3x]_\beta = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}.$$

$T(x^2) = (2x)(3 + x) + 2x^2 = 6x + 2x^2 + 2x^2 = 6x + 4x^2$

$$[6x + 4x^2]_\beta = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}.$$

$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}.$$

Compute UT on the basis. $UT(1) = U(2) = (2, 0, 2); [UT(1)]_\gamma = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

$UT(x) = U(3+3x) = (6,0,0); [UT(x)]_\gamma = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$. Here $a = 3, b = 3, c = 0$.

$UT(x^2) = U(6x+4x^2) = (6,4,-6); [UT(x^2)]_\gamma = \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix}$. Here $a = 0, b = 6, c = 4$.

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}.$$

Verification using Theorem 2.11 Theorem 2.11 states $[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta$.
Compute

$$[U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix} = [UT]_\beta^\gamma.$$

Thus the result is verified.

For 2: Given $h(x) = 3 - 2x + x^2$

$$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

Here, $a = 3, \quad b = -2, \quad c = 1, U(h(x)) = (a+b, c, a-b) = (3-2, 1, 3-(-2)) = (1,1,5)$.

$$[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}.$$

Verification using Theorem 2.14 Theorem 2.14 states $[U(h(x))]_\gamma = [U]_\beta^\gamma [h(x)]_\beta$.
Compute

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}.$$

This matches the direct computation.

Question 8: Prove Theorem 2.10. Now state and prove a more general result involving linear transformations whose domains are unequal to their codomains.

Theorem 2.10: Let $\mathbf{V}$ be a vector space. Let $T, U_1, U_2 \in \mathcal{L}(\mathbf{V})$. Then
(a). $T(U_1 + U_2) = TU_1 + TU_2$ and $(U_1 + U_2)T = U_1T + U_2T$
(b). $T(U_1U_2) = (TU_1)U_2$
(c). $TI = IT = T$
(d). $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$ for all scalars a
Xingjian's Solution:

Proof. **(a).** Let $v \in \mathbf{V}$. Then $\big(T(U_1+U_2)\big)(v) = T\big((U_1+U_2)(v)\big) = T\big(U_1(v)+U_2(v)\big)$. Since T is linear, $T\big(U_1(v)+U_2(v)\big) = T(U_1(v)) + T(U_2(v)) = \big(TU_1\big)(v) + \big(TU_2\big)(v)$. Thus

$$\big(T(U_1+U_2)\big)(v) = \big(TU_1+TU_2\big)(v) \quad \text{for all } v \in \mathbf{V},$$

so $T(U_1+U_2) = TU_1 + TU_2$.

Similarly,

$$\big((U_1+U_2)T\big)(v) = (U_1+U_2)\big(T(v)\big) = U_1(T(v)) + U_2(T(v)) = \big(U_1T\big)(v) + \big(U_2T\big)(v),$$

hence $(U_1+U_2)T = U_1T + U_2T$.

(b). Let $v \in \mathbf{V}$. Then by definition of composition, $\big(T(U_1U_2)\big)(v) = T\big((U_1U_2)(v)\big) = T\big(U_1(U_2(v))\big)$. On the other hand, $\big((TU_1)U_2\big)(v) = (TU_1)\big(U_2(v)\big) = T\big(U_1(U_2(v))\big)$. Both expressions are equal for all v, therefore $T(U_1U_2) = (TU_1)U_2$.

(c). Let $v \in \mathbf{V}$. Since $I(v) = v$, $(TI)(v) = T(I(v)) = T(v)$, so $TI = T$. Also, $(IT)(v) = I(T(v)) = T(v)$, so $IT = T$. Hence $TI = IT = T$.

(d). Let a be any scalar and $v \in \mathbf{V}$. Then $\big(a(U_1U_2)\big)(v) = a\,(U_1U_2)(v) = a\,U_1(U_2(v))$. Also, $\big((aU_1)U_2\big)(v) = (aU_1)\big(U_2(v)\big) = a\,U_1(U_2(v))$. Thus $a(U_1U_2) = (aU_1)U_2$.

Finally,

$$\big(U_1(aU_2)\big)(v) = U_1\big((aU_2)(v)\big) = U_1\big(a\,U_2(v)\big) = a\,U_1(U_2(v)),$$

using linearity of U_1. Hence $a(U_1U_2) = U_1(aU_2)$.

Therefore,

$$a(U_1U_2) = (aU_1)U_2 = U_1(aU_2) \quad \text{for all scalars } a.$$

□

Question 9: Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

Xingjian's Solution:

Proof. We construct linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ but $TU \neq T_0$. Let $T(x,y) = (0,x), \quad U(x,y) = (x,0)$.

Step 1: Compute UT. For any $(x,y) \in F^2$, $T(x,y) = (0,x)$. Applying U, $(UT)(x,y) = U(0,x) = (0,0)$. Hence $UT = T_0$.

Step 2: Compute TU. For any $(x,y) \in F^2$, $U(x,y) = (x,0)$. Applying T, $(TU)(x,y) = T(x,0) = (0,x)$. This is not the zero vector in general (for example when $x \neq 0$), therefore $TU \neq T_0$.

Step 3: Find matrices A and B such that $AB = O$ but $BA \neq O$.

Using the standard basis $e_1 = (1,0)$, $e_2 = (0,1)$:

For $U(x,y) = (x,0)$, $U(e_1) = (1,0), \quad U(e_2) = (0,0)$. Thus $A = [U] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

For $T(x,y) = (0,x)$, $T(e_1) = (0,1), T(e_2) = (0,0)$. Thus $B = [T] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Step 4: Verify the matrix products.

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O.$$

However,

$$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq O.$$

Therefore we have found matrices A and B such that $AB = O$ but $BA \neq O$. □

Question 11: Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

Xingjian's Solution:

Proof. We prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

$\Rightarrow$ Assume $T^2 = T_0$. Let $y \in R(T)$. Then by definition of range, there exists $x \in V$ such that $y = T(x)$. Apply T to y: $T(y) = T(T(x)) = T^2(x) = T_0(x) = 0$. Thus $y \in N(T)$. Since $y \in R(T)$ was arbitrary, we conclude $R(T) \subseteq N(T)$.

$\Leftarrow$ Assume $R(T) \subseteq N(T)$. Let $x \in V$ be arbitrary. Then $T(x) \in R(T)$ by definition of range. By the assumption $R(T) \subseteq N(T)$, we have $T(x) \in N(T)$, meaning $T(T(x)) = 0$. Equivalently, $T^2(x) = 0 \quad$ for all $x \in V$. Hence $T^2 = T_0$.

Therefore,

$$T^2 = T_0 \iff R(T) \subseteq N(T).$$

□

Question 12: Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

(a). Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

(b). Prove that if UT is onto, then U is onto. Must T also be onto?

(c). Prove that if U and T are both one-to-one and onto, then UT is also one-to-one and onto.

Xingjian's Solution:

Proof. (a). Assume UT is one-to-one. To show T is one-to-one, let $v_1, v_2 \in V$ and suppose $T(v_1) = T(v_2)$. Apply U to both sides: $U(T(v_1)) = U(T(v_2))$. That is, $(UT)(v_1) = (UT)(v_2)$. Since UT is one-to-one, it follows that $v_1 = v_2$. Hence T is one-to-one.

Must U also be one-to-one? No.

Counterexample. Let $V = \mathbb{R}$, $W = \mathbb{R}^2$, $Z = \mathbb{R}$. Define $T(t) = (t, 0), U(x, y) = x$. Then $(UT)(t) = U(t, 0) = t$, so UT is one-to-one. However, U is not one-to-one because

$$U(0, 0) = 0 = U(0, 1), \quad \text{but } (0, 0) \neq (0, 1).$$

(b). Assume UT is onto. To show U is onto, let $z \in Z$ be arbitrary. Since UT is onto, there exists $v \in V$ such that $(UT)(v) = z$. Let $w = T(v) \in W$. Then $U(w) = U(T(v)) = (UT)(v) = z$. Thus for every $z \in Z$ we found $w \in W$ with $U(w) = z$, so U is onto.

Must T also be onto? No.

Counterexample. Let $V = \mathbb{R}$, $W = \mathbb{R}^2$, $Z = \mathbb{R}$. Define $T(t) = (t, 0), U(x, y) = x$. Then, as above, $(UT)(t) = t$, so UT is onto $\mathbb{R}$. But T is not onto $\mathbb{R}^2$, since vectors such as $(0, 1)$ are not in the range of T.

(c). Assume T and U are both one-to-one and onto.

(one-to-one) Let $v_1, v_2 \in V$ and suppose $(UT)(v_1) = (UT)(v_2)$. Then $U(T(v_1)) = U(T(v_2))$. Since U is one-to-one, we get $T(v_1) = T(v_2)$. Since T is one-to-one, it follows that $v_1 = v_2$. Hence UT is one-to-one.

(onto) Let $z \in Z$ be arbitrary. Since U is onto, there exists $w \in W$ such that $U(w) = z$. Since T is onto, there exists $v \in V$ such that $T(v) = w$. Therefore, $(UT)(v) = U(T(v)) = U(w) = z$. Thus UT is onto.

Hence, if U and T are both one-to-one and onto, then UT is also one-to-one and onto. □

Question 13: Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\mathrm{tr}(A) = \mathrm{tr}(A^t)$.

Xingjian's Solution:

Proof. **(1) Prove** $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices. The (i, i)-entry of the matrix AB is

$$(AB)_{ii} = \sum_{j=1}^{n} a_{ij} b_{ji}.$$

Thus

$$\mathrm{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}.$$

Similarly, the (i, i)-entry of BA is

$$(BA)_{ii} = \sum_{j=1}^{n} b_{ij} a_{ji}.$$

Therefore

$$\mathrm{tr}(BA) = \sum_{i=1}^{n} (BA)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_{ji}.$$

Since multiplication in the field is commutative,

$$a_{ij} b_{ji} = b_{ji} a_{ij}.$$

Hence

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_{ji}.$$

Therefore $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(2). Prove $\mathrm{tr}(A) = \mathrm{tr}(A^t)$

Recall that the transpose of $A = (a_{ij})$ satisfies

$$(A^t)_{ij} = a_{ji}.$$

Thus the diagonal entries of A^t are

$$(A^t)_{ii} = a_{ii}.$$

Therefore

$$\mathrm{tr}(A^t) = \sum_{i=1}^{n}(A^t)_{ii} = \sum_{i=1}^{n} a_{ii} = \mathrm{tr}(A).$$

$$\boxed{\mathrm{tr}(AB) = \mathrm{tr}(BA), \qquad \mathrm{tr}(A) = \mathrm{tr}(A^t)}$$

□

Xingjian Xu
xingjianxu@ufl.edu
LIT 453
Webpage: https://people.clas.ufl.edu/xingjianxu/mas4105-26springs/
Office Hours: Tuesday, Periods 3 and 5 on LIT453;