Question 14: Let $\mathbf{V} = P(\mathbb{R})$, and for $j \geq 1$ define $T_j(f(x)) = f^{(j)}(x)$, where $f^{(j)}(x)($ is the jth derivative of $f(x)$. Prove that the set $\{T_1, T_2, \cdots, T_n\}$ is a linearly independent subset of $L(\mathbf{V})$ for any positive integer n.

Xingjian's Solution:

Proof. Let $\{T_1, \ldots, T_n\}$ be as defined and suppose we have a linear relation

$$a_1T_1 + a_2T_2 + \cdots + a_nT_n = 0 \quad \text{in } L(\mathbf{V}),$$

where $a_1, \ldots, a_n \in \mathbb{R}$ (or in the underlying field F). This means that the linear operator $L = \sum_{j=1}^{n} a_jT_j$ is the zero transformation on $\mathbf{V}$; i.e.,

$$L(f) = 0 \quad \text{for all } f \in \mathbf{V}.$$

We will show that this forces $a_1 = a_2 = \cdots = a_n = 0$.

For each $k = 1, 2, \ldots, n$, define the polynomial $f_k(x) = \frac{x^k}{k!}$. We first compute its derivatives at $x = 0$. For $j \leq k$, $f_k^{(j)}(x) = \frac{x^{k-j}}{(k-j)!}$, so

$$f_k^{(j)}(0) = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j < k. \end{cases}$$

If $j > k$, then $f_k^{(j)}(x) \equiv 0$, hence $f_k^{(j)}(0) = 0$. Thus for all $j \geq 1$,

$$f_k^{(j)}(0) = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

Now use the fact that L is the zero transformation. For each k,

$$0 = (Lf_k)(0) = \left(\sum_{j=1}^{n} a_jT_j(f_k)\right)(0) = \sum_{j=1}^{n} a_jf_k^{(j)}(0).$$

By the computation above, all terms vanish except when $j = k$, so

$$0 = \sum_{j=1}^{n} a_jf_k^{(j)}(0) = a_k.$$

Therefore $a_k = 0$ for each $k = 1, \ldots, n$.

Hence the only linear relation among $T_1, \ldots, T_n$ is the trivial one, so $\{T_1, \ldots, T_n\}$ is linearly independent in $L(\mathbf{V})$. □

Question 15: Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces, and let S be a subset of $\mathbf{V}$. Define $S^0 = T \in L(\mathbf{V}, \mathbf{W}) : T(x) = 0$ for all $x \in S$. Prove the following statements.

(a). S^0 is a subspace of $L(\mathbf{V}, \mathbf{W})$.

(b). If S_1 and S_2 are subsets of $\mathbf{V}$ and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

(c). If $\mathbf{V}_1$ and $\mathbf{V}_2$ are subspaces of $\mathbf{V}$, then $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

Xingjian's Solution:

Proof. **(a)** WTS S^0 is a subspace of $L(\mathbf{V}, \mathbf{W})$. First, note that the zero transformation $0 : \mathbf{V} \to \mathbf{W}$ defined by $0(x) = 0_{\mathbf{W}}$ for all $x \in \mathbf{V}$ belongs to S^0, because $0(x) = 0$ for all $x \in S$. Hence S^0 is nonempty.

Next, let $T_1, T_2 \in S^0$ and let $a \in F$ (the underlying field). We must show that $T_1+T_2 \in S^0$ and $aT_1 \in S^0$. Take any $x \in S$. Since $T_1, T_2 \in S^0$, we have $T_1(x) = 0$ and $T_2(x) = 0$. Then

$$(T_1 + T_2)(x) = T_1(x) + T_2(x) = 0 + 0 = 0,$$

so $T_1 + T_2$ vanishes on S and hence is in S^0.

Similarly,

$$(aT_1)(x) = a\,T_1(x) = a \cdot 0 = 0,$$

so aT_1 also vanishes on S and belongs to S^0.

Thus S^0 is closed under addition and scalar multiplication, and therefore it is a subspace of $L(\mathbf{V}, \mathbf{W})$.

(b) Suppose $S_1 \subseteq S_2$. Let $T \in S_2^0$. By definition, this means $T(x) = 0$ for all $x \in S_2$. Since $S_1 \subseteq S_2$, every $x \in S_1$ is also in S_2, so $T(x) = 0$ for all $x \in S_1$ as well. Hence $T \in S_1^0$. Thus $S_2^0 \subseteq S_1^0$.

(c)

First inclusion: $(\mathbf{V}_1 + \mathbf{V}_2)^0 \subseteq \mathbf{V}_1^0 \cap \mathbf{V}_2^0$.

Let $T \in (\mathbf{V}_1+\mathbf{V}_2)^0$. Then $T(x) = 0$ for all $x \in \mathbf{V}_1+\mathbf{V}_2$. In particular, since $\mathbf{V}_1 \subseteq \mathbf{V}_1+\mathbf{V}_2$ and $\mathbf{V}_2 \subseteq \mathbf{V}_1 + \mathbf{V}_2$, we have $T(v_1) = 0$ for all $v_1 \in \mathbf{V}_1$ and $T(v_2) = 0$ for all $v_2 \in \mathbf{V}_2$. Thus $T \in \mathbf{V}_1^0$ and $T \in \mathbf{V}_2^0$, so $T \in \mathbf{V}_1^0 \cap \mathbf{V}_2^0$.

Hence $(\mathbf{V}_1 + \mathbf{V}_2)^0 \subseteq \mathbf{V}_1^0 \cap \mathbf{V}_2^0$.

Second inclusion: $\mathbf{V}_1^0 \cap \mathbf{V}_2^0 \subseteq (\mathbf{V}_1 + \mathbf{V}_2)^0$.

Let $T \in \mathbf{V}_1^0 \cap \mathbf{V}_2^0$. Then T vanishes on both $\mathbf{V}_1$ and $\mathbf{V}_2$; that is, $T(v_1) = 0$ for all $v_1 \in \mathbf{V}_1$ and $T(v_2) = 0$ for all $v_2 \in \mathbf{V}_2$.

Now take any $x \in \mathbf{V}_1+\mathbf{V}_2$. By definition of the sum of subspaces, we can write $x = v_1+v_2$ with $v_1 \in \mathbf{V}_1$ and $v_2 \in \mathbf{V}_2$. Then, using linearity of T,

$$T(x) = T(v_1 + v_2) = T(v_1) + T(v_2) = 0 + 0 = 0.$$

Thus T vanishes on $\mathbf{V}_1 + \mathbf{V}_2$, so $T \in (\mathbf{V}_1 + \mathbf{V}_2)^0$.

We have shown both inclusions, so

$$(\mathbf{V}_1 + \mathbf{V}_2)^0 = \mathbf{V}_1^0 \cap \mathbf{V}_2^0.$$

□

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