Question 1: Label the following statements as true or false. Assume that $\mathbf{V}$ and $\mathbf{W}$ are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $T, U : \mathbf{V} \to \mathbf{W}$ are linear transformations.

(a). For any scalar a, $aT + U$ is a linear transformation from $\mathbf{V}$ to $\mathbf{W}$.
(b). $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies that $T = U$.
(c). If $m = \dim(\mathbf{V})$ and $n = \dim(\mathbf{W})$, then $[T]_\beta^\gamma$ is an $m \times n$ matrix.
(d). $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.
(e). $L(\mathbf{V}, \mathbf{W})$ is a vector space.
(f). $L(\mathbf{V}, \mathbf{W}) = L(\mathbf{W}, \mathbf{V})$.
Xingjian's Solution:

(a) **True.** For any scalar a and linear transformations $T, U : \mathbf{V} \to \mathbf{W}$, $(aT + U)(x) = aT(x) + U(x)$. For $x, y \in \mathbf{V}$ and $c \in \mathbb{R}$,

$$(aT+U)(x+y) = aT(x+y)+U(x+y) = a(Tx+Ty)+(Ux+Uy) = (aT+U)(x)+(aT+U)(y),$$

$$(aT + U)(cx) = aT(cx) + U(cx) = acT(x) + cU(x) = c(aT + U)(x).$$

Thus, $aT + U$ is linear.

(b) **True.** If $[T]_\beta^\gamma = [U]_\beta^\gamma$, then for each basis vector $v_j \in \beta$, the coordinate vectors of $T(v_j)$ and $U(v_j)$ relative to γ are equal. Hence $T(v_j) = U(v_j)$ for all basis vectors. By linearity, $T = U$ on V.

(c) **False.** If $\dim(V) = m$ and $\dim(W) = n$, then $[T]_\beta^\gamma$ has m columns (one for each basis vector of V) and n rows (coordinates in W). Hence it is an $n \times m$ matrix, not $m \times n$.

(d) **True.** For each basis vector v_j,$(T+U)(v_j) = T(v_j)+U(v_j)$. Taking coordinate vectors relative to γ gives $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.

(e) **True.** $L(\mathbf{V}, \mathbf{W})$ is closed under addition and scalar multiplication, and satisfies all vector space axioms. Therefore $L(\mathbf{V}, \mathbf{W})$ is a vector space.

(f) **False.** In general, $L(\mathbf{V}, \mathbf{W})$ and $L(\mathbf{W}, \mathbf{V})$ are different sets. There is no natural equality between them.

Question 2: Let β and γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.
Xingjian's Solution:
Compute: $T(e_1) = T(1, 0) = (2, 3, 1), \quad T(e_2) = T(0, 1) = (-1, 4, 0)$.
Thus

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}.$$

(d). $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (2a_2 + a_3, -a_1 + 4a_2 + 5a_3, a_1 + a_3)$.

Xingjian's Solution:
Compute: $T(e_1) = (0, -1, 1), \quad T(e_2) = (2, 4, 0), \quad T(e_3) = (1, 5, 1).$
Thus

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{bmatrix}.$$

(e). $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(a_1, a_2, \cdots, a_n) = (a_1, a_1, ..., a_1)$.
Xingjian's Solution:
Compute

$$T(e_1) = (1, 1, \ldots, 1), \qquad T(e_j) = 0 \text{ for } j \geq 2.$$

Thus,

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

(f). $T : \mathbb{R}^n \to \mathbb{R}^n$defined by $T(a_1, a_2, ..., a_n) = (a_n, a_{n-1}, ..., a_1)$.
Xingjian's Solution:
Compute

$$T(e_j) = e_{n+1-j}.$$

Thus

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Question 5: Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$\beta = \{1, x, x^2\}$, and $\gamma = \{1\}$.
(a). Define $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ by $T(A) = A^t$. Compute $[T]_\alpha$.
(b) Define

$$T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}) \text{ by } T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $'$ denotes differentiation. Compute $[T]_\beta^\alpha$.
(c) Define $T : M_{2\times 2}(F) \to F$ by $T(A) = \text{tr}(A)$. Compute $[T]_\alpha^\gamma$.
(d) Define $T : P_2(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$.
(e) If $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$, compute $[A]_\alpha$.

(f). If $f(x) = 3 - 6x + x^2$, compute $[f(x)]_\beta$.
(g). For $a \in F$, compute $[a]_\gamma$.
Xingjian's Solution:
We label the basis

$$\alpha = \left\{ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

the standard basis of $M_{2\times 2}(F)$.

(a) $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ given by $T(A) = A^t$.
We compute T on the basis α:

$$T(A_1) = A_1, \quad T(A_2) = A_3, \quad T(A_3) = A_2, \quad T(A_4) = A_4.$$

Writing each $T(A_j)$ in the basis α:

$$[T(A_1)]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(A_2)]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(A_3)]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(A_4)]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$[T]_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}, \quad \beta = \{1, x, x^2\},\ \alpha \text{ as above.}$$

First evaluate T on the basis β.

$$\begin{aligned} f(x) = 1: \quad & f'(x) = 0,\ f(1) = 1,\ f''(x) = 0 \\ & \Rightarrow T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0A_1 + 2A_2 + 0A_3 + 0A_4, \\ f(x) = x: \quad & f'(x) = 1,\ f(1) = 1,\ f''(x) = 0 \\ & \Rightarrow T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1A_1 + 2A_2 + 0A_3 + 0A_4, \\ f(x) = x^2: \quad & f'(x) = 2x,\ f'(0) = 0,\ f(1) = 1,\ f''(x) = 2 \\ & \Rightarrow T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 0A_1 + 2A_2 + 0A_3 + 2A_4. \end{aligned}$$

Therefore the coordinate columns in α are

$$[T(1)]_\alpha = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_\alpha = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x^2)]_\alpha = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix},$$

so

$$[T]_\beta^\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(c) $T : M_{2\times 2}(F) \to F$ given by $T(A) = \mathrm{tr}(A)$ and $\gamma = \{1\}$.
Evaluate on α:

$$T(A_1) = 1, \quad T(A_2) = 0, \quad T(A_3) = 0, \quad T(A_4) = 1.$$

In basis $\gamma = \{1\}$, the coordinates are scalars:

$$[T(A_1)]_\gamma = \begin{bmatrix} 1 \end{bmatrix}, \ [T(A_2)]_\gamma = \begin{bmatrix} 0 \end{bmatrix}, \ [T(A_3)]_\gamma = \begin{bmatrix} 0 \end{bmatrix}, \ [T(A_4)]_\gamma = \begin{bmatrix} 1 \end{bmatrix}.$$

Thus

$$[T]_\alpha^\gamma = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}.$$

(d) $T : P_2(\mathbb{R}) \to \mathbb{R}$ given by $T(f(x)) = f(2)$, with $\beta = \{1, x, x^2\}$ and $\gamma = \{1\}$.
Evaluate:

$$T(1) = 1, \quad T(x) = 2, \quad T(x^2) = 4.$$

So in basis γ,

$$[T(1)]_\gamma = \begin{bmatrix} 1 \end{bmatrix}, \ [T(x)]_\gamma = \begin{bmatrix} 2 \end{bmatrix}, \ [T(x^2)]_\gamma = \begin{bmatrix} 4 \end{bmatrix}.$$

Hence

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}.$$

(e) If

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix},$$

write A in the basis α:

$$A = 1A_1 + (-2)A_2 + 0A_3 + 4A_4.$$

So

$$[A]_\alpha = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix}.$$

(f) If $f(x) = 3 - 6x + x^2$ and $\beta = \{1, x, x^2\}$, then

$$f(x) = 3 \cdot 1 + (-6) \cdot x + 1 \cdot x^2,$$

so

$$[f(x)]_\beta = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}.$$

(g) For $a \in F$ and $\gamma = \{1\}$ a basis of F, we have

$$a = a \cdot 1,$$

thus

$$[a]_\gamma = \begin{bmatrix} a \end{bmatrix}.$$

Question 6: Reprove Theorem 2.7: Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over a field F, and let $T, U : \mathbf{V} \to \mathbf{W}$ be linear.

(a) For all $a \in F, aT + U$ is linear.

(b) Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformations from $\mathbf{V}$ to $\mathbf{W}$ is a vector space over F.

Xingjian's Solution:

Proof. (a). Define $S : \mathbf{V} \to \mathbf{W}$ by $S(v) = (aT + U)(v) := aT(v) + U(v)$. Let $v, w \in \mathbf{V}$ and $\lambda, \mu \in F$. Then

$$\begin{aligned} S(\lambda v + \mu w) &= aT(\lambda v + \mu w) + U(\lambda v + \mu w) \\ &= a\big(\lambda T(v) + \mu T(w)\big) + \big(\lambda U(v) + \mu U(w)\big) \\ &= \lambda a T(v) + \mu a T(w) + \lambda U(v) + \mu U(w) \\ &= \lambda\big(aT(v) + U(v)\big) + \mu\big(aT(w) + U(w)\big) \\ &= \lambda S(v) + \mu S(w). \end{aligned}$$

Thus $aT + U$ is linear.

(b). Let $\mathcal{L}(\mathbf{V}, \mathbf{W}) = \{\, T : \mathbf{V} \to \mathbf{W} \mid T \text{ is linear} \,\}$. Define addition and scalar multiplication by $(T + U)(v) := T(v) + U(v), \quad (aT)(v) := aT(v)$.

By part (a), $\mathcal{L}(\mathbf{V}, \mathbf{W})$ is closed under these operations. The zero transformation 0 defined by $0(v) = 0_{\mathbf{W}}$ is linear and satisfies $T + 0 = T$. For each T, the map $-T$ defined by $(-T)(v) = -T(v)$ is linear and satisfies $T + (-T) = 0$.

All other vector space axioms (associativity, commutativity, distributive laws, and scalar identities) follow from the corresponding properties in $\mathbf{W}$. Therefore $\mathcal{L}(\mathbf{V}, \mathbf{W})$ is a vector space over F. □

Question 7: Reprove Theorem 2.8: Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let $T, U : \mathbf{V} \to \mathbf{W}$ be linear transformations. Then

(a). $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.

(b). $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars a.

Xingjian's Solution:

Proof. (a). Let $v \in \mathbf{V}$. Then $[(T+U)(v)]_\gamma = [T(v)+U(v)]_\gamma$. Since coordinate maps preserve addition, $[T(v)+U(v)]_\gamma = [T(v)]_\gamma + [U(v)]_\gamma$. Using the definition of matrix representation, $[T(v)]_\gamma = [T]^\gamma_\beta[v]_\beta$, $[U(v)]_\gamma = [U]^\gamma_\beta[v]_\beta$. Hence $[(T+U)(v)]_\gamma = \big([T]^\gamma_\beta + [U]^\gamma_\beta\big)[v]_\beta$. But by definition, $[(T+U)(v)]_\gamma = [T+U]^\gamma_\beta[v]_\beta$.

Since this holds for all v, we conclude $[T+U]^\gamma_\beta = [T]^\gamma_\beta + [U]^\gamma_\beta$.

(b). Let $v \in \mathbf{V}$. Then $[(aT)(v)]_\gamma = [aT(v)]_\gamma = a[T(v)]_\gamma$. Using matrix representation, $[T(v)]_\gamma = [T]^\gamma_\beta[v]_\beta$. Thus $[(aT)(v)]_\gamma = a[T]^\gamma_\beta[v]_\beta = (a[T]^\gamma_\beta)[v]_\beta$.

But $[(aT)(v)]_\gamma = [aT]^\gamma_\beta[v]_\beta$.

Since this holds for all v, $[aT]^\gamma_\beta = a[T]^\gamma_\beta$. □

Question 10: Let $\mathbf{V}$ be a vector space with the ordered basis $\beta = \{v_1, v_2, \cdots, v_n\}$. Define $v_0 = 0$. By Theorem 2.6 (p. 72), there exists a linear transformation $T : \mathbf{V} \to \mathbf{V}$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \cdots, n$. Compute $[T]_\beta$.

Xingjian's Solution:

Proof. □

For each basis vector v_j we write $T(v_j)$ in coordinates relative to β.

For $j = 1$:

$$T(v_1) = v_1 + v_0 = v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n,$$

so the first column of $[T]_\beta$ is $\big[1, 0, 0, \ldots, 0\big]^T$.

For $j \geq 2$:

$$T(v_j) = v_j + v_{j-1} = 1 \cdot v_{j-1} + 1 \cdot v_j,$$

so the j-th column has a 1 in the $(j-1)$-st row (coefficient of v_{j-1}), a 1 in the j-th row (coefficient of v_j), and 0 elsewhere.

Thus the matrix $[T]_\beta$ has the form

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

an $n \times n$ matrix with 1's on the main diagonal and on the superdiagonal just above it, and 0's elsewhere.

Question 13: Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces, and let T and U be nonzero linear transformations from $\mathbf{V}$ into $\mathbf{W}$. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $L(\mathbf{V}, \mathbf{W})$.

Xingjian's Solution:

To show $\{T, U\}$ is linearly independent, we must show that if $aT + bU = 0$ in $L(\mathbf{V}, \mathbf{W})$ for scalars a, b, then $a = b = 0$.

Suppose $aT + bU = 0$. This means that for every $v \in \mathbf{V}$, $(aT + bU)(v) = aT(v) + bU(v) = 0$. Thus, $aT(v) = -bU(v)$.

For each $v \in \mathbf{V}$, the vector $aT(v)$ lies in $R(T)$ and $-bU(v)$ lies in $R(U)$, so $aT(v) \in R(T) \cap R(U)$. By hypothesis, $R(T) \cap R(U) = \{0\}$, hence $aT(v) = 0 \quad$ for all $v \in \mathbf{V}$. So aT is the zero transformation. Since T is nonzero, this forces $a = 0$. With $a = 0$, the equation $aT + bU = 0$ becomes $bU = 0$. Thus bU is the zero transformation, and since U is nonzero we must have $b = 0$.

Therefore the only linear relation $aT + bU = 0$ is the trivial one $a = b = 0$. Hence $\{T, U\}$ is linearly independent in $L(\mathbf{V}, \mathbf{W})$.

Question 16: Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces such that $\dim(V) = \dim(W)$, and let $T : \mathbf{V} \to \mathbf{W}$ be linear. Show that there exist ordered bases β and γ for $\mathbf{V}$ and $\mathbf{W}$, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

Xingjian's Solution:

Let $r = \text{rank}(T) = \dim R(T)$. Pick a basis $\{w_1, \ldots, w_r\}$ of the subspace $R(T) \subseteq \mathbf{W}$ and extend it to a basis $\gamma = \{w_1, \ldots, w_r, w_{r+1}, \ldots, w_n\}$ of $\mathbf{W}$. For each $i = 1, \ldots, r$, since $w_i \in R(T)$ there exists $v_i \in \mathbf{V}$ with $T(v_i) = w_i$.

Let $\{u_{r+1}, \ldots, u_n\}$ be a basis of $\ker T$. (By the dimension Theorem, $\dim \ker T = n - r$.) Now form the ordered set

$$\beta = \{v_1, \ldots, v_r, u_{r+1}, \ldots, u_n\}.$$

We claim β is a basis of $\mathbf{V}$: - the vectors are n in number, - $\{u_{r+1}, \ldots, u_n\}$ spans $\ker T$, - the images $T(v_1), \ldots, T(v_r)$ form a basis of $R(T)$, - and $R(T) \cap \ker T = \{0\}$, so the sets combine to give a basis of $\mathbf{V}$.

By construction, $T(v_i) = w_i \quad$ for $i = 1, \ldots, r$, and $T(u_j) = 0 \quad$ for $j = r+1, \ldots, n$. Relative to the basis γ, we have $[T(v_i)]_\gamma = e_i \quad (i = 1, \ldots, r)$, where e_i is the i-th standard basis vector in $\mathbb{F}^n$, and $[T(u_j)]_\gamma = 0 \quad (j = r+1, \ldots, n)$.

Therefore, the columns of $[T]_\beta^\gamma$ are:

$$\text{column } i = \begin{cases} e_i, & 1 \le i \le r, \\ 0, & r+1 \le i \le n. \end{cases}$$

Thus

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & 0 \\ & & 1 & & & 0 \\ & & & 0 & & 0 \\ & & & & \ddots & 0 \\ 0 & & & & & 0 \end{bmatrix},$$

an $n \times n$ diagonal matrix with r entries equal to 1 on the diagonal and $n - r$ entries equal to 0. Hence there exist bases β of $\mathbf{V}$ and γ of $\mathbf{W}$ such that $[T]_\beta^\gamma$ is diagonal, as required.

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