Question 19: Give an example of distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$.

Xingjian's Solution:

Proof. Take $V = W = \mathbb{R}^2$ and define

$$T(x, y) = (x, y) \quad \text{(the identity map)},$$

$$U(x, y) = (2x, 2y) \quad \text{(multiplication by 2)}.$$

Both T and U are linear, and clearly $T \neq U$ because, for example,

$$T(1, 0) = (1, 0) \neq (2, 0) = U(1, 0).$$

Null spaces: If $T(x, y) = (0, 0)$, then $(x, y) = (0, 0)$. So $N(T) = \{(0, 0)\}$. If $U(x, y) = (0, 0)$, then $(2x, 2y) = (0, 0)$, so again $(x, y) = (0, 0)$. Thus $N(U) = \{(0, 0)\}$.

Hence

$$N(T) = N(U) = \{(0, 0)\}.$$

Ranges: For T, we have $R(T) = \mathbb{R}^2$ since T is the identity. For U, given any $(a, b) \in \mathbb{R}^2$, we can write

$$(a, b) = U\Big(\frac{a}{2}, \frac{b}{2}\Big),$$

so every vector in $\mathbb{R}^2$ is in the image of U. Thus $R(U) = \mathbb{R}^2$.

Therefore

$$R(T) = R(U) = \mathbb{R}^2,$$

and we have an example of distinct linear transformations with the same kernel and range. □

Question 20: Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces with subspaces $\mathbf{V}_1$ and $\mathbf{W}_1$, respectively. If $T : \mathbf{V} \to \mathbf{W}$ is linear, prove that $T(V_1)$ is a subspace of W and that $\{x \in \mathbf{V} : T(x) \in W_1\}$ is a subspace of $\mathbf{V}$.

Xingjian's Solution:

Proof. (1) Let $T : V \to W$ be linear and V_1 a subspace of V. We prove that $T(V_1)$ is a subspace of W.

First, $T(V_1) \subseteq W$ by definition, so we only need to check the subspace properties:

Nonempty: Since V_1 is a subspace, it contains 0_V. Because T is linear, $T(0_V) = 0_W$, hence $0_W \in T(V_1)$, so $T(V_1)$ is nonempty.

Closed under addition: Let $y_1, y_2 \in T(V_1)$. Then there exist $x_1, x_2 \in V_1$ such that $y_1 = T(x_1)$ and $y_2 = T(x_2)$. Since V_1 is a subspace, $x_1 + x_2 \in V_1$, and

$$y_1 + y_2 = T(x_1) + T(x_2) = T(x_1 + x_2) \in T(V_1).$$

So $T(V_1)$ is closed under addition.

Closed under scalar multiplication: Let $y \in T(V_1)$ and $c \in F$. Then $y = T(x)$ for some $x \in V_1$. Since V_1 is a subspace, $cx \in V_1$, and

$$cy = cT(x) = T(cx) \in T(V_1).$$

Thus $T(V_1)$ is a subspace of W.

(2) Now define

$$U := \{x \in V : T(x) \in W_1\}.$$

We show that U is a subspace of V.

Nonempty: Since W_1 is a subspace of W, we have $0_W \in W_1$. Because T is linear, $T(0_V) = 0_W \in W_1$, so $0_V \in U$. Hence U is nonempty.

Closed under addition: Let $x_1, x_2 \in U$. Then $T(x_1) \in W_1$ and $T(x_2) \in W_1$. Since W_1 is a subspace,

$$T(x_1) + T(x_2) \in W_1.$$

But $T(x_1 + x_2) = T(x_1) + T(x_2)$ by linearity, so $T(x_1 + x_2) \in W_1$, and therefore $x_1 + x_2 \in U$.

Closed under scalar multiplication: Let $x \in U$ and $c \in F$. Then $T(x) \in W_1$. Since W_1 is a subspace,

$$cT(x) \in W_1.$$

By linearity, $T(cx) = cT(x)$, so $T(cx) \in W_1$, hence $cx \in U$.

Thus U is closed under addition and scalar multiplication, and contains 0, so U is a subspace of V. □

Question 21: Let F be a field. A *sequence* in F is a function $\sigma : \mathbb{N} \to F$, which we write as

$$\sigma = (a_1, a_2, a_3, \dots).$$

Let $\mathbf{V}$ be the set of all sequences in F that have only finitely many nonzero terms. Define addition and scalar multiplication componentwise:

$$(a_1, a_2, a_3, \dots) + (b_1, b_2, b_3, \dots) = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3, \dots),$$

$$c(a_1, a_2, a_3, \dots) = (ca_1,\ ca_2,\ ca_3, \dots), \quad c \in F.$$

With these operations, $\mathbf{V}$ is a vector space.

Define linear operators $T, U : \mathbf{V} \to \mathbf{V}$ by

$$T(a_1, a_2, a_3, \dots) = (a_2, a_3, a_4, \dots),$$

$$U(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots).$$

The operator T is called the *left shift operator*, and U is called the *right shift operator*.

(a) Prove that T and U are linear.

Xingjian's Solution:

Proof. We show that both T and U satisfy

$$L(\mathbf{a}+\mathbf{b}) = L(\mathbf{a}) + L(\mathbf{b}), \qquad L(c\mathbf{a}) = c\,L(\mathbf{a})$$

for all sequences $\mathbf{a}, \mathbf{b} \in V$ and scalars $c \in F$.

Write $\mathbf{a} = (a_1, a_2, a_3, \dots)$ and $\mathbf{b} = (b_1, b_2, b_3, \dots)$.

Linearity of T:

First, additivity:

$$T(\mathbf{a}+\mathbf{b}) = T(a_1+b_1,\ a_2+b_2,\ a_3+b_3, \dots) = (a_2+b_2,\ a_3+b_3, \dots).$$

On the other hand,

$$T(\mathbf{a}) + T(\mathbf{b}) = (a_2, a_3, \dots) + (b_2, b_3, \dots) = (a_2+b_2,\ a_3+b_3, \dots).$$

So $T(\mathbf{a}+\mathbf{b}) = T(\mathbf{a}) + T(\mathbf{b})$.

Next, homogeneity:

$$T(c\mathbf{a}) = T(ca_1, ca_2, ca_3, \dots) = (ca_2, ca_3, \dots) = c(a_2, a_3, \dots) = c\,T(\mathbf{a}).$$

Thus T is linear.

Linearity of U:

Additivity:

$$U(\mathbf{a}+\mathbf{b}) = U(a_1+b_1,\ a_2+b_2, \dots) = (0,\ a_1+b_1,\ a_2+b_2, \dots).$$

While

$$U(\mathbf{a}) + U(\mathbf{b}) = (0, a_1, a_2, \dots) + (0, b_1, b_2, \dots) = (0,\ a_1+b_1,\ a_2+b_2, \dots).$$

Hence $U(\mathbf{a}+\mathbf{b}) = U(\mathbf{a}) + U(\mathbf{b})$.

Homogeneity:

$$U(c\mathbf{a}) = U(ca_1, ca_2, \dots) = (0, ca_1, ca_2, \dots) = c(0, a_1, a_2, \dots) = c\,U(\mathbf{a}).$$

Therefore U is linear as well. □

(b) Prove that T is onto, but not one-to-one.

Xingjian's Solution:

Proof. **Not one-to-one:**

We show that T has a nontrivial kernel. Compute

$$T(1, 0, 0, 0, \dots) = (0, 0, 0, \dots) = 0.$$

Also

$$T(0, 0, 0, \dots) = 0.$$

So at least two different vectors are mapped to 0. Equivalently, the kernel of T contains all sequences of the form $(a_1, 0, 0, 0, \dots)$, which is more than just the zero sequence. Therefore T is not one-to-one.

Onto:

Let $\mathbf{b} = (b_1, b_2, b_3, \dots) \in V$ be any sequence with only finitely many nonzero entries. Define

$$\mathbf{a} := (0, b_1, b_2, b_3, \dots).$$

Since $\mathbf{b}$ has only finitely many nonzero terms, so does $\mathbf{a}$, and thus $\mathbf{a} \in V$.

Now compute

$$T(\mathbf{a}) = T(0, b_1, b_2, b_3, \dots) = (b_1, b_2, b_3, \dots) = \mathbf{b}.$$

Thus for every $\mathbf{b} \in V$ there exists $\mathbf{a} \in V$ with $T(\mathbf{a}) = \mathbf{b}$, so T is onto. □

(c) Prove that U is one-to-one, but not onto.

Xingjian's Solution:

Proof. **One-to-one:**

Suppose $U(\mathbf{a}) = U(\mathbf{b})$ for sequences

$$\mathbf{a} = (a_1, a_2, a_3, \dots), \quad \mathbf{b} = (b_1, b_2, b_3, \dots).$$

Then

$$U(\mathbf{a}) = (0, a_1, a_2, a_3, \dots), \quad U(\mathbf{b}) = (0, b_1, b_2, b_3, \dots).$$

Equality of these sequences means

$$0 = 0, \quad a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3, \dots$$

so $a_n = b_n$ for all n, i.e., $\mathbf{a} = \mathbf{b}$. Therefore U is one-to-one.

Not onto:

Observe that any sequence in the range of U has the form

$$U(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots),$$

so its first component is always 0.

Consider the sequence

$$\mathbf{c} = (1, 0, 0, 0, \dots) \in V.$$

This has first component $1 \neq 0$. There is no sequence $\mathbf{a}$ such that $U(\mathbf{a}) = \mathbf{c}$, because $U(\mathbf{a})$ always starts with 0.

Hence $\mathbf{c}$ is not in the image of U, so U is not onto. □

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