Xingjian comment: Before the solution, I'd like to introduce my motivation for what we are learning now in chapter 2.

We have already developed the theory of vector spaces. However, in mathematics and in nature, we do not encounter just a single vector space. There are many different vector spaces: some are familiar and concrete, while others may be abstract or difficult to understand directly.

To study and compare such spaces, we need a way to relate them. One natural idea is to use functions that map a known vector space into another space. Through such mappings, we hope to transfer structural information. For example, to send a basis of a known vector space to a set of vectors in another space, and to represent vectors in the new space in a systematic way. Ideally, the algebraic calculations in the original space should correspond exactly to the algebraic calculations in the target space.

However, not every function between vector spaces is suitable for this purpose. To see what we require, consider a simple example. The real numbers $\mathbb{R}$ form a vector space with addition and scalar multiplication. For instance,

$$1 + 1 = 2, 2 \times 3 = 6$$

Suppose there is a function f that maps $\mathbb{R}$ into another vector space. A natural question arises: does

$$f(1) + f(1) = f(2) \quad \text{and} \quad f(2)f(3) = f(6)?$$

In other words, does the function preserve addition and scalar multiplication? Only when a function preserves these equalities does it truly respect the algebraic structure of a vector space. Such structure-preserving functions are actually linear transformations you learned right now. and we are trying to research these function maps in this chapter to match this algebraic structure between two vector spaces.

Question 1: Label the following statements as true or false. In each part, $\mathbf{V}$ and $\mathbf{W}$ are finite-dimensional vector spaces (over F), and T is a function from $\mathbf{V}$ to $\mathbf{W}$.

(a). If T is linear, then T preserves sums and scalar products.

(b). If $T(x + y) = T(x) + T(y)$, then T is linear.

(c). T is one-to-one if and only if the only vector x such that $T(x) = 0$ is $x = 0$.

(d). If T is linear, then $T(0_V) = 0_W$.

(e). If T is linear, then $\text{nullity}(T) + \text{rank}(T) = \dim(W)$.

(f). If T is linear, then T carries linearly independent subsets of $\mathbf{V}$ onto linearly independent subsets of $\mathbf{W}$.

(g). If $T, U : \mathbf{V} \to \mathbf{W}$ are both linear and agree on a basis for $\mathbf{V}$, then $T = U$.

(h). Given $x_1, x_2 \in \mathbf{V}$ and $y_1, y_2 \in \mathbf{W}$, there exists a linear transformation $T : \mathbf{V} \to \mathbf{W}$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

Xingjian's Solution:

(a) **True.** If T is linear, then for all $x, y \in \mathbf{V}$ and $c \in F$, $T(x+y) = T(x)+T(y), T(cx) = cT(x)$, so T preserves sums and scalar multiples by definition.

(b) **False.** The condition alone does not guarantee linearity; we also need $T(cx) = cT(x)$ for all scalars c.

(c) **True.** By definition, T is one-to-one (injective) if $T(x) = T(y)$ implies $x = y$. This is equivalent to saying that the only vector mapped to 0 is $x = 0$, i.e., $N(T) = \ker(T) = \{x \in \mathbf{V} : T(x) = 0\} = \{0\}$.

(d) **True.** If T is linear, then $T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$, so subtracting $T(0_V)$ from both sides gives $T(0_V) = 0_W$.

(e) **False.** The dimension theorem says $\text{nullity}(T) + \text{rank}(T) = \dim(\mathbf{V})$, not $\dim(\mathbf{W})$ in general.

(f) **False.** A linear transformation T does not necessarily carry linearly independent sets to linearly independent sets. This property holds if T is one-to-one. If T is not injective (i.e., $\ker(T) \neq \{0\}$), a linearly independent set can map to a linearly *dependent* set.

(g) **True.** Let $\{v_1, \ldots, v_n\}$ be a basis for $\mathbf{V}$. If T and U are linear and agree on this basis, i.e., $T(v_i) = U(v_i)$ for all i, then for any $x \in \mathbf{V}$ we can write $x = \sum_{i=1}^{n} a_i v_i$, so

$$T(x) = T\Big(\sum_{i=1}^{n} a_i v_i\Big) = \sum_{i=1}^{n} a_i T(v_i) = \sum_{i=1}^{n} a_i U(v_i) = U(x).$$

Hence $T = U$.

(h) **False.** In general, this cannot always be done. For example, if $x_2 = cx_1$ for some scalar c, linearity would force

$$T(x_2) = T(cx_1) = cT(x_1) = cy_1.$$

So we must have $y_2 = cy_1$. If we choose y_1, y_2 that do not satisfy this relation, no linear T with $T(x_1) = y_1$ and $T(x_2) = y_2$ can exist. Thus the statement is false without additional conditions.

For Question 3, 5, prove that T is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

Question 3: $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

Xingjian's Solution: *Xingjian's Solution:*

Part 1: Prove that T is a linear transformation. For any $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Then

$$T\big((a_1, a_2) + (b_1, b_2)\big) = T(a_1 + b_1, a_2 + b_2) = (a_1 + b_1 + a_2 + b_2,\ 0,\ 2(a_1 + b_1) - (a_2 + b_2)).$$

On the other hand,

$$T(a_1, a_2) + T(b_1, b_2) = (a_1 + a_2, 0, 2a_1 - a_2) + (b_1 + b_2, 0, 2b_1 - b_2)$$

$$= (a_1 + b_1 + a_2 + b_2,\ 0,\ 2(a_1 + b_1) - (a_2 + b_2)),$$

which agrees with $T((a_1, a_2) + (b_1, b_2))$. Hence T preserves sums.

For scalar multiples,

$$T\big(c(a_1, a_2)\big) = T(ca_1, ca_2) = (ca_1 + ca_2, 0, 2ca_1 - ca_2) = c(a_1 + a_2, 0, 2a_1 - a_2) = cT(a_1, a_2).$$

Thus T preserves scalar multiples, and T is linear.

Part 2: Find a basis for $N(T)$. The null space is $N(T) = \{(a_1, a_2) \in \mathbb{R}^2 : T(a_1, a_2) = (0, 0, 0)\}$. So we must solve

$$(a_1 + a_2, 0, 2a_1 - a_2) = (0, 0, 0),$$

i.e.,

$$a_1 + a_2 = 0, \qquad 2a_1 - a_2 = 0.$$

From $a_1 + a_2 = 0$ we get $a_2 = -a_1$. Substitute into $2a_1 - a_2 = 0$:

$$2a_1 - (-a_1) = 3a_1 = 0 \quad \Rightarrow \quad a_1 = 0 \Rightarrow a_2 = 0.$$

Thus $N(T) = \{(0, 0)\}$. The only vector in the kernel is the zero vector, so the null space has dimension 0; a basis is the empty set. Therefore $\text{nullity}(T) = 0$.

Part 3: Find a basis for $R(T)$. Consider the images of the standard basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$:

$$T(e_1) = T(1, 0) = (1, 0, 2), \qquad T(e_2) = T(0, 1) = (1, 0, -1).$$

We claim that

$$R(T) = \text{span}\{(1, 0, 2), (1, 0, -1)\}.$$

Indeed, for any $(a_1, a_2) \in \mathbb{R}^2$,

$$T(a_1, a_2) = a_1T(1, 0) + a_2T(0, 1) = a_1(1, 0, 2) + a_2(1, 0, -1),$$

so every image lies in this span. Conversely, the span of these two vectors is contained in $R(T)$, so they generate $R(T)$.

To check linear independence, suppose

$$c_1(1, 0, 2) + c_2(1, 0, -1) = (0, 0, 0).$$

This gives the system

$$c_1 + c_2 = 0, \qquad 2c_1 - c_2 = 0.$$

From $c_1 + c_2 = 0$ we get $c_2 = -c_1$. Substitute into $2c_1 - c_2 = 0$:

$$2c_1 - (-c_1) = 3c_1 = 0 \Rightarrow c_1 = 0 \Rightarrow c_2 = 0.$$

Thus the two vectors are linearly independent. Therefore,

$$\dim R(T) = 2, \quad \text{and a basis for } R(T) \text{ is } \{(1, 0, 2), (1, 0, -1)\}.$$

Hence $\text{rank}(T) = 2$.

Part 4: Verify the dimension theorem. The domain is $\mathbb{R}^2$, so $\dim(\mathbb{R}^2) = 2$. We found $\text{nullity}(T) = 0, \text{rank}(T) = 2$, and indeed $\text{nullity}(T) + \text{rank}(T) = 0 + 2 = 2 = \dim(\mathbb{R}^2)$. This agrees with the dimension theorem.

Part 5: One-to-one and onto.

Since $N(T) = \{0\}$, T is one-to-one(injective). However, the codomain is $\mathbb{R}^3$, which has dimension 3, while $\text{rank}(T) = 2$. Thus $R(T)$ is a proper subspace of $\mathbb{R}^3$, so T is *not* onto.

Question 5: $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.

Xingjian's Solution:

Part 1: Prove that T is a linear transformation. For each $f, g \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$T(cf+g) = x(cf+g)+(cf+g)' = cxf+xg+cf'+g' = c(xf+f')+(xg+g') = cT(f)+T(g).$$

Thus T is a linear transformation.

Part 2: Find a basis for $N(T)$. Let $f(x) = ax^2+bx+c \in P_2(\mathbb{R})$. Then $f'(x) = 2ax+b$, and

$$T(f(x)) = x(ax^2+bx+c)+(2ax+b) = ax^3+bx^2+cx+2ax+b = ax^3+bx^2+(c+2a)x+b.$$

For f to be in the kernel, we need $T(f(x)) \equiv 0$ as a polynomial, so all coefficients must be zero:

$$a = 0, \quad b = 0, \quad c + 2a = 0, \quad b = 0.$$

From $a = 0$ and $b = 0$, we get $c + 2a = c = 0$. Thus $a = b = c = 0$ and $f(x) \equiv 0$.

Hence

$$N(T) = \{0\} \subset P_2(\mathbb{R}),$$

so $\text{nullity}(T) = 0$ and the null space has the empty set as a basis.

Part 3: Find a basis for $R(T)$. Consider the standard basis of $P_2(\mathbb{R})$: $\{1, x, x^2\}$.

$$T(1) = x \cdot 1 + (1)' = x + 0 = x,$$
$$T(x) = x \cdot x + (x)' = x^2 + 1,$$
$$T(x^2) = x \cdot x^2 + (x^2)' = x^3 + 2x.$$

Therefore,

$$R(T) = \text{span}\{x,\ x^2+1,\ x^3+2x\} \subset P_3(\mathbb{R}).$$

We show that these three polynomials are linearly independent. Suppose

$$\alpha x + \beta(x^2+1) + \gamma(x^3+2x) \equiv 0$$

for some scalars α, β, γ. Expanding gives

$$\gamma x^3 + \beta x^2 + (\alpha + 2\gamma)x + \beta = 0$$

as a polynomial. Hence the coefficients all vanish:

$$\gamma = 0, \quad \beta = 0, \quad \alpha + 2\gamma = 0, \quad \beta = 0.$$

From $\gamma = 0$ and $\beta = 0$, we get $\alpha + 2 \cdot 0 = \alpha = 0$. Thus

$$\alpha = \beta = \gamma = 0,$$

so the three polynomials are linearly independent. Therefore

$$\dim R(T) = 3, \quad \text{and a basis for } R(T) \text{ is } \{x,\ x^2 + 1,\ x^3 + 2x\}.$$

Hence $\text{rank}(T) = 3$.

Step 4: Verify the dimension theorem. We know $\dim P_2(\mathbb{R}) = 3$. We found $\text{nullity}(T) = 0, \text{rank}(T) = 3$, and so $\text{nullity}(T) + \text{rank}(T) = 0 + 3 = 3 = \dim P_2(\mathbb{R})$, which verifies the dimension theorem.

Step 5: One-to-one and onto. Since $N(T) = \{0\}$, T is one-to-one. The codomain $P_3(\mathbb{R})$ has dimension 4. Because $\text{rank}(T) = 3 < 4$, the range is a proper subspace of $P_3(\mathbb{R})$, so T is *not* onto.

Question 7: Prove properties 1, 2, 3, and 4 on page 65. Let $\mathbf{V}$ and $\mathbf{W}$ be two vector spaces. And $T : \mathbf{V} \to \mathbf{W}$ be the linear transformation. We need to prove the following.

(a). If T is linear, then $T(0) = 0$.

(b). T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in \mathbf{V}$ and $c \in F$.

(c). If T is linear, then $T(x - y) = T(x) - T(y)$ for all $x, y \in \mathbf{V}$.

(d). T is linear if and only if, for $x_1, x_2, \cdots, x_n \in V$ and $a_1, a_2, \cdots, a_n \in F$, we have

$$T\big(\sum_{i=1}^{n} a_i x_i\big) = \sum_{i=1}^{n} a_i T(x_i)$$

.

Xingjian's Solution:

Proof. **(a).** Let 0 denote the zero vector in $\mathbf{V}$. Since $0 = 0 + 0$, we have $T(0) = T(0 + 0) = T(0) + T(0)$ by additivity. Subtract $T(0)$ from both sides in $\mathbf{W}$ to get $T(0) = 0$.

(b).($\Rightarrow$) Suppose T is linear. Then $T(cx + y) = T(cx) + T(y) = cT(x) + T(y)$, so the identity holds.

($\Leftarrow$) Conversely, assume that for all $x, y \in \mathbf{V}$ and $c \in F$, $T(cx + y) = cT(x) + T(y)$. First show $T(0) = 0$. Take $x = 0$, $y = 0$, and $c = 1$:

$$T(1 \cdot 0 + 0) = T(0) = 1 \cdot T(0) + T(0) = 2T(0),$$

so $T(0) = 2T(0)$, hence $T(0) = 0$.

Now check additivity. For any $x, y \in \mathbf{V}$, set $c = 1$:

$$T(x + y) = T(1 \cdot x + y) = 1 \cdot T(x) + T(y) = T(x) + T(y).$$

Next, check homogeneity. For any $x \in \mathbf{V}$ and scalar $c \in F$, take $y = 0$:

$$T(cx) = T(cx + 0) = cT(x) + T(0) = cT(x) + 0 = cT(x).$$

Thus T preserves sums and scalar multiples, so T is linear.

(c). Note that $x - y = x + (-1)y$. By linearity,

$$T(x - y) = T(x + (-1)y) = T(x) + T((-1)y) = T(x) + (-1)T(y) = T(x) - T(y).$$

(d).($\Rightarrow$) Suppose T is linear. We prove the formula by induction on n.

For $n = 1$,

$$T(a_1x_1) = a_1T(x_1),$$

which holds by homogeneity.

Assume the statement holds for $n - 1 \geq 1$, i.e.,

$$T\Big(\sum_{i=1}^{n-1} a_ix_i\Big) = \sum_{i=1}^{n-1} a_iT(x_i).$$

Then for n we write

$$\sum_{i=1}^{n} a_ix_i = \Big(\sum_{i=1}^{n-1} a_ix_i\Big) + a_nx_n,$$

and use linearity:

$$T\Big(\sum_{i=1}^{n} a_ix_i\Big) = T\Big(\sum_{i=1}^{n-1} a_ix_i + a_nx_n\Big) = T\Big(\sum_{i=1}^{n-1} a_ix_i\Big) + T(a_nx_n)$$

$$= \sum_{i=1}^{n-1} a_iT(x_i) + a_nT(x_n) = \sum_{i=1}^{n} a_iT(x_i).$$

Thus the formula holds for all n by induction.

($\Leftarrow$) Conversely, assume that for all finite collections $\{x_i\}$ and scalars $\{a_i\}$,

$$T\Big(\sum_{i=1}^{n} a_ix_i\Big) = \sum_{i=1}^{n} a_iT(x_i).$$

In particular, for $n = 2$, letting $x_1 = x$, $x_2 = y$ and $a_1 = c$, $a_2 = d$, we get

$$T(cx + dy) = cT(x) + dT(y)$$

for all $x, y \in \mathbf{V}$ and $c, d \in F$. Taking $d = 0$ gives $T(cx) = cT(x)$ (homogeneity); taking $c = d = 1$ gives $T(x + y) = T(x) + T(y)$ (additivity). Hence T is linear. □

Question 9: In this exercise, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. For each of the following parts, state why T is not linear.

(a). $T(a_1, a_2) = (1, a_2)$.

Xingjian's Solution:

Proof. If T were linear, we must have $T(0,0) = (0,0)$, because any linear transformation sends the zero vector to the zero vector.

However,

$$T(0,0) = (1,0) \neq (0,0).$$

Thus T does not send the zero vector to the zero vector, so T cannot be linear. □

(b). $T(a_1, a_2) = (a_1, a_1^2)$.
Xingjian's Solution:

Proof. For a linear transformation T, we must have $T(c\mathbf{v}) = cT(\mathbf{v})$ for all scalars c and vectors $\mathbf{v}$.

Take $\mathbf{v} = (1,0)$ and $c = 2$. Then

$$T(1,0) = (1,1), \quad \text{so} \quad 2T(1,0) = 2(1,1) = (2,2).$$

On the other hand,

$$T(2,0) = (2, 2^2) = (2,4).$$

Since $(2,2) \neq (2,4)$, we have $T(2\mathbf{v}) \neq 2T(\mathbf{v})$, so T fails homogeneity and is not linear. □

(c). $T(a_1, a_2) = (\sin(a_1), 0)$.
Xingjian's Solution: *Xingjian's Solution:*

Proof. Again, linearity requires $T(c\mathbf{v}) = cT(\mathbf{v})$ for all scalars c and vectors $\mathbf{v}$.

Take $\mathbf{v} = (1,0)$ and $c = 2$. Then

$$T(1,0) = (\sin 1, 0), \quad \text{so} \quad 2T(1,0) = (2\sin 1, 0).$$

But

$$T(2,0) = (\sin 2, 0).$$

Since $\sin 2 \neq 2 \sin 1$ in general, we have $T(2\mathbf{v}) \neq 2T(\mathbf{v})$. Thus T does not satisfy homogeneity and is not linear. □

(d) $T(a_1, a_2) = (|a_1|, a_2)$
Xingjian's Solution:

Proof. For any linear transformation T, we must have $T(-\mathbf{v}) = -T(\mathbf{v})$.

Take $\mathbf{v} = (1,0)$. Then

$$T(1,0) = (|1|, 0) = (1,0).$$

If T were linear, then

$$T(-1,0) = -T(1,0) = (-1,0).$$

But by the definition of T,

$$T(-1,0) = (|-1|, 0) = (1,0) \neq (-1,0).$$

So $T(-\mathbf{v}) \neq -T(\mathbf{v})$ and T is not linear. □

(e) $T(a_1, a_2) = (a_1 + 1, a_2)$
Xingjian's Solution:

Proof. A linear transformation must send the zero vector to the zero vector.
However,

$$T(0,0) = (0+1,0) = (1,0) \neq (0,0).$$

Thus T does not send 0 to 0, so T cannot be linear. □

Question 11: Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?
Xingjian's Solution:

Proof. For this question, I use step 1-step 5 to calculate the linear transformation.
Step 1: Show that $(1,1)$ and $(2,3)$ form a basis of $\mathbb{R}^2$.
Suppose

$$c_1(1,1) + c_2(2,3) = (0,0).$$

Then

$$c_1 + 2c_2 = 0, \qquad c_1 + 3c_2 = 0.$$

Subtracting the first equation from the second gives

$$c_2 = 0.$$

Substituting back gives $c_1 = 0$.
Thus the two vectors are linearly independent. Since $\mathbb{R}^2$ has dimension 2, they form a basis.

Step 2: Define T explicitly.
Since $\{(1,1),(2,3)\}$ is a basis, every vector $(x,y) \in \mathbb{R}^2$ can be written uniquely as

$$(x,y) = c_1(1,1) + c_2(2,3).$$

Define T by

$$T(x,y) := c_1(1,0,2) + c_2(1,-1,4),$$

where c_1, c_2 are the unique scalars satisfying

$$(x,y) = c_1(1,1) + c_2(2,3).$$

We must now verify that this definition is well-defined and linear.

Step 3: Solve for c_1, c_2.
From

$$(x,y) = c_1(1,1) + c_2(2,3),$$

we obtain

$$c_1 + 2c_2 = x, \qquad c_1 + 3c_2 = y.$$

Subtracting the first equation from the second gives

$$c_2 = y - x.$$

Substituting into $c_1 + 2c_2 = x$:

$$c_1 = x - 2(y - x) = 3x - 2y.$$

Thus

$$c_1 = 3x - 2y, \qquad c_2 = y - x.$$

Step 4: Obtain an explicit formula for $T(x, y)$.
Now substitute into the definition:

$$T(x, y) = (3x - 2y)(1, 0, 2) + (y - x)(1, -1, 4).$$

Compute each component.
First coordinate:

$$(3x - 2y) + (y - x) = 2x - y.$$

Second coordinate:

$$0 + (-(y - x)) = x - y.$$

Third coordinate:

$$2(3x - 2y) + 4(y - x) = 6x - 4y + 4y - 4x = 2x.$$

Therefore

$$\boxed{T(x, y) = (2x - y,\ x - y,\ 2x).}$$

Step 5: Verify the required values.
Check $T(1, 1)$:

$$T(1, 1) = (2 - 1,\ 1 - 1,\ 2) = (1, 0, 2).$$

Check $T(2, 3)$:

$$T(2, 3) = (4 - 3,\ 2 - 3,\ 4) = (1, -1, 4).$$

Thus the formula satisfies the required conditions.

Step 6: Prove linearity.
The formula

$$T(x, y) = (2x - y,\ x - y,\ 2x)$$

is linear in x and y (each coordinate is a linear polynomial in x, y).
For $(x_1, y_1), (x_2, y_2)$:

$$T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2)$$

$$= (2(x_1 + x_2) - (y_1 + y_2),\ (x_1 + x_2) - (y_1 + y_2),\ 2(x_1 + x_2)),$$

which equals

$$T(x_1, y_1) + T(x_2, y_2).$$

Similarly,

$$T(c(x, y)) = cT(x, y).$$

Thus T is linear.

Step 7: Compute $T(8, 11)$.
Using the explicit formula,

$$T(8, 11) = (2 \cdot 8 - 11,\ 8 - 11,\ 2 \cdot 8) = (16 - 11,\ -3,\ 16) = \boxed{(5, -3, 16)}.$$

□

Question 12: Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, -6) = (2, 1)$?
Xingjian's Solution:

Proof. our goal is to check whether the given values are consistent with linearity. Note that $(-2, 0, -6) = -2(1, 0, 3)$. If T were linear, then

$$T(-2, 0, -6) = T\big(-2(1, 0, 3)\big) = -2\,T(1, 0, 3).$$

Using the given value $T(1, 0, 3) = (1, 1)$, linearity would force

$$T(-2, 0, -6) = -2(1, 1) = (-2, -2).$$

However, the problem requires $T(-2, 0, -6) = (2, 1)$, which is not equal to $(-2, -2)$.

This contradicts what linearity would require. Therefore, no linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ can satisfy both conditions simultaneously. □

Question 13: Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces, let $T : \mathbf{V} \to \mathbf{W}$ be linear, and let $\{w_1, w_2, ..., w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, ..., v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \cdots, k$, then S is linearly independent
Xingjian's Solution:

Proof. Suppose there exists scalars $a_1, \ldots, a_k$ s.t. $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0$, then we want to show $a_1 = \cdots = a_k = 0$.

Apply the linear map T to both sides:

$$T(a_1v_1 + \cdots + a_kv_k) = T(0).$$

Using linearity and the fact that $T(0) = 0$, we get

$$a_1T(v_1) + \cdots + a_kT(v_k) = 0,$$

i.e.

$$a_1w_1 + \cdots + a_kw_k = 0.$$

But by assumption, $\{w_1, \ldots, w_k\}$ is linearly independent in $R(T) \subseteq \mathbf{W}$, so the only solution to

$$a_1 w_1 + \cdots + a_k w_k = 0$$

is

$$a_1 = a_2 = \cdots = a_k = 0.$$

Hence $\{v_1, \ldots, v_k\}$ is linearly independent in $\mathbf{V}$. □

Question 14: Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces and $T : \mathbf{V} \to \mathbf{W}$ be linear.

(a). Prove that T is one-to-one if and only if T carries linearly independent subsets of $\mathbf{V}$ onto linearly independent subsets of $\mathbf{W}$.

(b). Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c). Suppose $\beta = \{v_1, v_2, \cdots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis for $\mathbf{W}$.

Xingjian's Solution:

Proof. **(a).** ($\Rightarrow$) Assume T is one-to-one. Let $S = \{v_1, \ldots, v_k\}$ be a linearly independent subset of $\mathbf{V}$. We show that $T(S) = \{T(v_1), \ldots, T(v_k)\}$ is linearly independent in $\mathbf{W}$.

Suppose

$$a_1 T(v_1) + \cdots + a_k T(v_k) = 0$$

for scalars a_i. By linearity,

$$T(a_1 v_1 + \cdots + a_k v_k) = 0.$$

Since T is one-to-one, its kernel is $\{0\}$, so

$$a_1 v_1 + \cdots + a_k v_k = 0.$$

But S is linearly independent, hence $a_1 = \cdots = a_k = 0$. Therefore $T(S)$ is linearly independent.

($\Leftarrow$) Conversely, assume that T carries every linearly independent subset of $\mathbf{V}$ onto a linearly independent subset of $\mathbf{W}$. We prove that T is one-to-one.

Suppose, by contradiction, that T is not one-to-one. Then there exists a nonzero vector $v \in \mathbf{V}$ such that $T(v) = 0$. The set $\{v\}$ is linearly independent in $\mathbf{V}$, but its image is $\{T(v)\} = \{0\}$, which is linearly *dependent*. This contradicts the assumption that T sends linearly independent sets to linearly independent sets.

Hence T must be one-to-one.

(b). Now assume T is one-to-one and let $S \subset V$ be any subset.

($\Rightarrow$) If S is linearly independent, then by part (a), $T(S)$ is linearly independent.

($\Leftarrow$) Conversely, suppose $T(S)$ is linearly independent. If S were linearly dependent, then there would exist distinct $v_1, \ldots, v_k \in S$ and scalars a_i, not all zero, such that

$$a_1 v_1 + \cdots + a_k v_k = 0.$$

Apply T:

$$a_1 T(v_1) + \cdots + a_k T(v_k) = T(0) = 0.$$

This gives a nontrivial linear relation among the vectors in $T(S)$, contradicting that $T(S)$ is linearly independent. Thus S must be linearly independent.

Therefore S is linearly independent if and only if $T(S)$ is linearly independent.

(c). Assume $\beta = \{v_1, \ldots, v_n\}$ is a basis for $\mathbf{V}$, and T is linear, one-to-one, and onto.

We show $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a basis for $\mathbf{W}$.

Linear independence: Since T is one-to-one, by part (a) (or (b)), the image of a linearly independent set under T is linearly independent. Because β is a basis, it is linearly independent in $\mathbf{V}$, so $T(\beta)$ is linearly independent in $\mathbf{W}$.

Spanning: Let $w \in \mathbf{W}$ be arbitrary. Since T is onto, there exists some $v \in \mathbf{V}$ such that $T(v) = w$. Because β is a basis, we can write

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Applying T and using linearity,

$$w = T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n).$$

Thus w is a linear combination of the vectors in $T(\beta)$. Hence $T(\beta)$ spans $\mathbf{W}$.

Since $T(\beta)$ is both linearly independent and spanning, it is a basis of $\mathbf{W}$. □

Question 15: Recall the definition of $P(\mathbb{R})$ on page 10. Define $T : P(\mathbb{R}) \to P(\mathbb{R})$ by

$$T(f(x)) = \int_0^x f(t)dt.$$

Prove that T linear and one-to-one, but not onto.

Xingjian's Solution:

Proof. **Part 1: Linearity.** Let $f, g \in P(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$T(cf + g)(x) = \int_0^x (cf(t) + g(t))\, dt = c\int_0^x f(t)\, dt + \int_0^x g(t)\, dt = cT(f)(x) + T(g)(x).$$

Thus $T(cf + g) = cT(f) + T(g)$ for all f, g, so T is linear.

Part 2: One-to-one(injective). Suppose $T(f) = 0$ (the zero polynomial). This means

$$\int_0^x f(t)\, dt = 0 \quad \text{for all } x.$$

The left-hand side is a polynomial in x. Since $P(\mathbb{R})$ is a space of polynomials and we are allowed to use basic calculus, we differentiate both sides with respect to x:

$$\frac{d}{dx}\left(\int_0^x f(t)\, dt\right) = f(x) = 0.$$

So $f(x) = 0$ for all x, i.e., f is the zero polynomial. Thus the kernel of T consists only of the zero polynomial, and T is one-to-one.

Part 3: Not onto. Note that, for any f,

$$T(f)(0) = \int_0^0 f(t)\,dt = 0.$$

So every polynomial in the range of T has constant term 0. For example, a polynomial like $p(x) = 1$ (or any polynomial with nonzero constant term) cannot be written as $T(f)$ for any $f \in P(\mathbb{R})$.

Hence T is not onto $P(\mathbb{R})$. □

Question 16: Let $T : P(\mathbb{R}) \to P(\mathbb{R})$ be defined by $T(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.

Xingjian's Solution:

Proof. **Part 1: Onto.**

Let $g(x) \in P(\mathbb{R})$ be any polynomial. We want $f(x) \in P(\mathbb{R})$ such that $f'(x) = g(x)$.

Define

$$f(x) := \int_0^x g(t)\,dt.$$

Since g is a polynomial, its integral is also a polynomial, so $f \in P(\mathbb{R})$. By the Fundamental Theorem of Calculus,

$$T(f)(x) = f'(x) = g(x).$$

Thus every polynomial g in the codomain is the derivative of some polynomial f, so T is onto.

Part 2: Not one-to-one.

Consider any constant polynomial, say $c(x) = C$ (with $C \neq 0$). Then

$$T(c)(x) = c'(x) = 0.$$

Also, $T(0)(x) = 0$. Thus both c and 0 are mapped to the same polynomial, so the kernel contains more than just the zero polynomial.

Equivalently, the kernel is

$$N(T) = \ker(T) = \{f \in P(\mathbb{R}) : f'(x) = 0\} = \{\text{constant polynomials}\},$$

which is a nontrivial subspace. Therefore T is not one-to-one. □

Question 17: Let V and W be finite-dimensional vector spaces and $T : \mathbf{V} \to \mathbf{W}$ be linear. (a). Prove that if $\dim(V) < \dim(W)$, then T cannot be onto. (b). Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

Xingjian's Solution:

Proof. **(a).** Suppose $\dim(V) < \dim(W)$ and, by contradiction, that T is onto. If T is onto, its range is all of W, hence $\text{rank}(T) = \dim(W)$. By dimension theorem,

$$\dim(V) = \text{rank}(T) + \text{nullity}(T) \geq \text{rank}(T) = \dim(W).$$

Thus $\dim(V) \geq \dim(W)$, which contradicts the assumption $\dim(V) < \dim(W)$.

Therefore, T cannot be onto when $\dim(V) < \dim(W)$.

(b). Suppose $\dim(V) > \dim(W)$ and, by contradiction, that T is one-to-one. Then the kernel of T is $\{0\}$, so

$$\text{nullity}(T) = 0.$$

By dimension theorem,

$$\dim(V) = \text{rank}(T) + \text{nullity}(T) = \text{rank}(T).$$

since a subspace of W cannot have dimension larger than W, then

$$\dim(V) = \text{rank}(T) \leq \dim(W),$$

which contradicts the assumption $\dim(V) > \dim(W)$.

Hence T cannot be one-to-one when $\dim(V) > \dim(W)$. □

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