Special Question from PhD qualifying exam:[similar to Theorem 1.9 in class but not the same, the most different part is S is not necessarily finite, it can have infinite cardinality] Let V be a finite-dimensional vector space over a field F, and let $S \subseteq V$ such that

$$\text{Span}(S) = V.$$

Then there exists a subset $B \subseteq S$ such that B is a basis for V.

Xingjian's Solution:

Proof. Since V is finite-dimensional, there exists a finite set S of V and $\text{Span}(S) = V$, we can choose finitely many vectors

$$v_1, v_2, \ldots, v_m \in S$$

such that

$$\text{Span}\{v_1, v_2, \ldots, v_m\} = V.$$

If the set $\{v_1, \ldots, v_m\}$ is linearly independent, then it is already a basis of V and is contained in S. In this case, we may take $B = \{v_1, \ldots, v_m\}$, and the proof is complete.

Otherwise, $\{v_1, \ldots, v_m\}$ is linearly dependent. Then one of the vectors can be written as a linear combination of the others. Hence there exists an index k such that

$$v_k \in \text{Span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_m\}.$$

Removing v_k does not change the span, so

$$\text{Span}\{v_1, \ldots, v_m\} = \text{Span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_m\}.$$

If the remaining set is linearly independent, then it is a basis of V. If not, we repeat the same process: remove a vector that is a linear combination of the others. Each removal strictly decreases the number of vectors while preserving the span.

Since the original set is finite, this process must terminate after finitely many steps. Eventually, we obtain a subset

$$B \subseteq \{v_1, \ldots, v_m\} \subseteq S$$

that is linearly independent and still spans V.

By definition, B is a basis of V. Therefore, there exists a subset $B \subseteq S$ that is a basis for V. □

Question 7: The vectors $u_1 = (2, -3, 1), u_2 = (1, 4, -2), u_3 = (-8, 12, -4), u_4 = (1, 37, -17)$ and $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

Xingjian's Solution:

Proof. First, $u_1 = (2, -3, 1) \neq 0$, so the set $\{u_1\}$ is linearly independent.

Next, check whether u_2 is a multiple of u_1. If $u_2 = \lambda u_1$, we would have

$$(1, 4, -2) = \lambda(2, -3, 1) = (2\lambda, -3\lambda, \lambda).$$

From the first component, $\lambda = \frac{1}{2}$, but then the second component would be $-3 \cdot \frac{1}{2} = -\frac{3}{2} \neq 4$, a contradiction. Hence u_2 is not a multiple of u_1, so $\{u_1, u_2\}$ is linearly independent.

Now consider $u_3 = (-8, 12, -4)$. Notice that

$$-4u_1 = -4(2, -3, 1) = (-8, 12, -4) = u_3,$$

so u_3 is a scalar multiple of u_1 and therefore adds no new direction. We *do not* include u_3 in our basis.

Next, test $u_4 = (1, 37, -17)$. We ask whether u_4 is in the span of $\{u_1, u_2\}$; i.e., do there exist scalars a, b such that

$$au_1 + bu_2 = u_4?$$

This is the same as solving

$$a(2, -3, 1) + b(1, 4, -2) = (1, 37, -17).$$

Componentwise, we get the system

$$\begin{cases} 2a + b = 1, \\ -3a + 4b = 37, \\ a - 2b = -17. \end{cases}$$

From the first equation, $b = 1 - 2a$. Substitute into the third:

$$a - 2(1 - 2a) = -17 \implies a - 2 + 4a = -17 \implies 5a = -15 \implies a = -3.$$

Then $b = 1 - 2(-3) = 1 + 6 = 7$. Check in the second equation:

$$-3(-3) + 4(7) = 9 + 28 = 37,$$

so the system is consistent. Thus

$$u_4 = -3u_1 + 7u_2,$$

so u_4 is in $\text{span}\{u_1, u_2\}$ and is not needed for a basis.

Finally, test $u_5 = (-3, -5, 8)$. Suppose u_5 were a linear combination of u_1 and u_2:

$$au_1 + bu_2 = u_5,$$

i.e.

$$a(2, -3, 1) + b(1, 4, -2) = (-3, -5, 8).$$

This leads to the system

$$\begin{cases} 2a + b = -3, \\ -3a + 4b = -5, \\ a - 2b = 8. \end{cases}$$

From $2a + b = -3$ we get $b = -3 - 2a$. Substitute into $a - 2b = 8$:

$$a - 2(-3 - 2a) = 8 \implies a + 6 + 4a = 8 \implies 5a + 6 = 8 \implies 5a = 2 \implies a = \frac{2}{5}.$$

Then $b = -3 - 2 \cdot \frac{2}{5} = -3 - \frac{4}{5} = -\frac{19}{5}$. Check in the second equation:

$$-3a + 4b = -3 \cdot \frac{2}{5} + 4 \cdot \left(-\frac{19}{5}\right) = -\frac{6}{5} - \frac{76}{5} = -\frac{82}{5} \neq -5.$$

So the system is inconsistent; there are no a, b such that $au_1 + bu_2 = u_5$. Thus u_5 is not in $\text{span}\{u_1, u_2\}$, and $\{u_1, u_2, u_5\}$ is linearly independent.

We have found three linearly independent vectors in $\mathbb{R}^3$, so they form a basis. Therefore, one possible subset that is a basis is $\{u_1, u_2, u_5\}$. □

Question 14: Find bases for the following subspaces of $\mathbf{F}^5$:

$$\mathbf{W_1} = \left\{(a_1, a_2, a_3, a_4, a_5) \in \mathbf{F}^5 : a_1 - a_3 - a_4 = 0\right\}$$

and

$$\mathbf{W_2} = \left\{(a_1, a_2, a_3, a_4, a_5) \in \mathbf{F}^5 : a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\right\}.$$

What are the dimension of $\mathbf{W}_1$ and $\mathbf{W}_2$?

Xingjian's Solution: *Xingjian's Solution:*

Proof. **(1) Basis and dimension of $\mathbf{W}_1$.**

The condition $a_1 - a_3 - a_4 = 0$ can be rewritten as

$$a_1 = a_3 + a_4.$$

So a general vector in $\mathbf{W}_1$ has the form

$$(a_1, a_2, a_3, a_4, a_5) = (a_3 + a_4,\ a_2,\ a_3,\ a_4,\ a_5).$$

Let us introduce free parameters

$$s = a_2, \quad t = a_3, \quad u = a_4, \quad v = a_5.$$

Then every vector in $\mathbf{W}_1$ can be written as

$$(a_1, a_2, a_3, a_4, a_5) = (t + u,\ s,\ t,\ u,\ v).$$

We now separate this into a linear combination:

$$(t + u,\ s,\ t,\ u,\ v) = s(0, 1, 0, 0, 0) + t(1, 0, 1, 0, 0) + u(1, 0, 0, 1, 0) + v(0, 0, 0, 0, 1).$$

Thus every element of $\mathbf{W}_1$ is a linear combination of the four vectors

$$w_1 = (0, 1, 0, 0, 0), \quad w_2 = (1, 0, 1, 0, 0), \quad w_3 = (1, 0, 0, 1, 0), \quad w_4 = (0, 0, 0, 0, 1).$$

So $\{w_1, w_2, w_3, w_4\}$ spans $\mathbf{W}_1$.

To check linear independence, suppose

$$c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4 = 0.$$

That is,

$$c_1(0,1,0,0,0) + c_2(1,0,1,0,0) + c_3(1,0,0,1,0) + c_4(0,0,0,0,1) = (0,0,0,0,0).$$

Adding componentwise gives

$$(c_2 + c_3,\ c_1,\ c_2,\ c_3,\ c_4) = (0,0,0,0,0),$$

so

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0.$$

Thus the only linear relation is the trivial one, and the set is linearly independent.

Therefore

$$\{(0,1,0,0,0),\ (1,0,1,0,0),\ (1,0,0,1,0),\ (0,0,0,0,1)\}$$

is a basis for $\mathbf{W}_1$. Since this basis has 4 vectors, we have

$$\dim \mathbf{W}_1 = 4.$$

(2) Basis and dimension of $\mathbf{W}_2$.

The conditions for $\mathbf{W}_2$ are

$$a_2 = a_3 = a_4 \quad \text{and} \quad a_1 + a_5 = 0.$$

Write $a_2 = a_3 = a_4 = y$ and $a_1 = x$, so $a_5 = -x$. Then a general vector in $\mathbf{W}_2$ has the form

$$(a_1, a_2, a_3, a_4, a_5) = (x,\ y,\ y,\ y,\ -x).$$

We can factor this as a linear combination:

$$(x,\ y,\ y,\ y,\ -x) = x(1,0,0,0,-1) + y(0,1,1,1,0).$$

Thus every vector in $\mathbf{W}_2$ is a linear combination of

$$v_1 = (1,0,0,0,-1), \qquad v_2 = (0,1,1,1,0).$$

So $\{v_1, v_2\}$ spans $\mathbf{W}_2$.

To check linear independence, suppose

$$\alpha v_1 + \beta v_2 = 0.$$

That is,

$$\alpha(1,0,0,0,-1) + \beta(0,1,1,1,0) = (0,0,0,0,0),$$

so

$$(\alpha,\ \beta,\ \beta,\ \beta,\ -\alpha) = (0,0,0,0,0).$$

This implies

$$\alpha = 0, \quad \beta = 0.$$

Hence $\{v_1, v_2\}$ is linearly independent.

Therefore

$$\{(1,0,0,0,-1),\ (0,1,1,1,0)\}$$

is a basis for $\mathbf{W}_2$, and since it has 2 vectors,

$$\dim \mathbf{W}_2 = 2.$$

□

Question 23: Let $v_1, v_2, \ldots, v_k, v$ be vectors in a vector space V, and define

$$W_1 = \text{span}\{v_1, \ldots, v_k\}, \qquad W_2 = \text{span}\{v_1, \ldots, v_k, v\}.$$

(a) Find necessary and sufficient conditions on v such that $\dim(W_1) = \dim(W_2)$.

(b) State and prove a relationship involving $\dim(W_1)$ and $\dim(W_2)$ in the case $\dim(W_1) \neq \dim(W_2)$.

Xingjian's Solution:

Proof. (a) Note that $W_1 \subseteq W_2$ always, because

$$\{v_1, \ldots, v_k\} \subseteq \{v_1, \ldots, v_k, v\}.$$

So W_2 is obtained from W_1 by possibly adding the new direction v.

We claim that

$$\dim(W_1) = \dim(W_2) \quad \Longleftrightarrow \quad v \in W_1.$$

($\Rightarrow$) Assume $\dim(W_1) = \dim(W_2)$. Since $W_1 \subseteq W_2$ and the two spaces have the same dimension, they must be equal:

$$W_1 = W_2.$$

But $v \in W_2$ by definition (v is in the spanning set for W_2), hence $v \in W_1$.

($\Leftarrow$) Conversely, assume $v \in W_1$. Then v is already a linear combination of $v_1, \ldots, v_k$, so adding v to the spanning set does not change the span:

$$\text{span}\{v_1, \ldots, v_k, v\} = \text{span}\{v_1, \ldots, v_k\} = W_1.$$

Thus $W_2 = W_1$, and so $\dim(W_1) = \dim(W_2)$.

This proves part (a).

(b) From part (a) we know that if $\dim(W_1) \neq \dim(W_2)$, then $v \notin W_1$. In this case v is not in the span of $\{v_1, \ldots, v_k\}$, so the set

$$\{v_1, \ldots, v_k, v\}$$

is linearly independent if and only if $\{v_1, \ldots, v_k\}$ is linearly independent.

In particular, if we start with a basis $\{w_1, \ldots, w_r\}$ of W_1 (so $r = \dim W_1$), then $v \notin W_1$ means that v is not a linear combination of $\{w_1, \ldots, w_r\}$, so the set

$$\{w_1, \ldots, w_r, v\}$$

is linearly independent. Its span is W_2, so this set is a basis of W_2 and

$$\dim(W_2) = r + 1 = \dim(W_1) + 1.$$

Therefore, if $\dim(W_1) \neq \dim(W_2)$, then necessarily

$$\dim(W_2) = \dim(W_1) + 1.$$

Combining with part (a), we can summarize:

$$\dim(W_2) = \begin{cases} \dim(W_1), & v \in W_1, \\ \dim(W_1) + 1, & v \notin W_1. \end{cases}$$

□

Question 24: Let $f(x)$ be a polynomial of degree n in $\mathbf{P}_n(\mathbb{R})$. Prove that for any $g(x) \in \mathbf{P}_n(\mathbb{R})$ there exist scalars $c_0, c_1, \ldots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x),$$

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$.

Xingjian's Solution:

Proof. We want to show that every polynomial $g \in \mathbf{P}_n(\mathbb{R})$ can be expressed as a linear combination of

$$f, f', f'', \ldots, f^{(n)}.$$

In other words, WTS $\{f, f', \ldots, f^{(n)}\}$ is a basis for $\mathbf{P}_n(\mathbb{R})$.

Since $f(x)$ has degree n, we can write

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, \qquad a_n \neq 0.$$

Then

$$\deg f = n, \quad \deg f' = n - 1, \quad \deg f'' = n - 2, \quad \ldots, \quad \deg f^{(n-1)} = 1, \quad \deg f^{(n)} = 0.$$

Moreover,

$$f^{(n)}(x) = n! a_n,$$

a nonzero constant polynomial. Thus all of $f, f', \ldots, f^{(n)}$ are nonzero and have distinct degrees.

Suppose we have a linear relation

$$c_0 f(x) + c_1 f'(x) + \cdots + c_n f^{(n)}(x) = 0$$

for all $x \in \mathbb{R}$, where $c_0, \ldots, c_n$ are scalars.

Look at the degrees: the polynomial f has degree n, and all other $f^{(k)}$ have degree at most $n-1$. Thus the only way the degree-n term can vanish is if $c_0 = 0$, because the leading term x^n can come only from f, not from its derivatives.

Once $c_0 = 0$, the highest-degree term remaining comes from f', which has degree $n-1$. By the same reasoning, its coefficient must be zero, so $c_1 = 0$. Continuing this argument step by step, we get

$$c_0 = c_1 = \cdots = c_n = 0.$$

Thus $\{f, f', \ldots, f^{(n)}\}$ is linearly independent.

The space $\mathbf{P}_n(\mathbb{R})$ has dimension $n+1$, and we have found $n+1$ linearly independent polynomials $f, f', \ldots, f^{(n)}$. Therefore this set is a basis for $\mathbf{P}_n(\mathbb{R})$.

By the definition of basis, every $g \in \mathbf{P}_n(\mathbb{R})$ can be written uniquely as a linear combination

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x)$$

for some scalars $c_0, \ldots, c_n$.

This is exactly what we needed to prove. □

Question 25: Let V, W, and Z be as in Exercise 21 of Section 1.2. If V and W are vector spaces over F of dimensions m and n, determine the dimension of Z.

Xingjian's Solution:

Proof. In Exercise 21 of Section 1.2, the space Z is defined as the Cartesian product

$$Z = V \times W = \{(v, w) : v \in V,\ w \in W\},$$

with addition and scalar multiplication defined componentwise:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2,\ w_1 + w_2), \qquad \lambda(v, w) = (\lambda v,\ \lambda w).$$

It was shown there that Z is a vector space over F.

We are given that $\dim V = m$ and $\dim W = n$. Let $\{v_1, \ldots, v_m\}$ be a basis for V, and $\{w_1, \ldots, w_n\}$ be a basis for W.

Consider the following set of vectors in Z:

$$\mathcal{B} = \{(v_1, 0), \ldots, (v_m, 0),\ (0, w_1), \ldots, (0, w_n)\}.$$

We claim that $\mathcal{B}$ is a basis for Z.

Spanning. Take any $(v, w) \in Z$. Since $\{v_1, \ldots, v_m\}$ is a basis of V, we can write

$$v = a_1 v_1 + \cdots + a_m v_m$$

for some scalars $a_1, \dots, a_m \in F$. Similarly, since $\{w_1, \dots, w_n\}$ is a basis of W, we can write

$$w = b_1 w_1 + \cdots + b_n w_n$$

for some scalars $b_1, \dots, b_n \in F$.

Then in Z we have

$$(v, w) = (v, 0) + (0, w) = \sum_{i=1}^{m} a_i (v_i, 0) + \sum_{j=1}^{n} b_j (0, w_j).$$

Thus every element of Z is a linear combination of the vectors in $\mathcal{B}$, so $\mathcal{B}$ spans Z.

Linear independence. Suppose

$$\sum_{i=1}^{m} \alpha_i (v_i, 0) + \sum_{j=1}^{n} \beta_j (0, w_j) = (0, 0)$$

for some scalars $\alpha_i, \beta_j \in F$. Then

$$\left(\sum_{i=1}^{m} \alpha_i v_i,\ \sum_{j=1}^{n} \beta_j w_j \right) = (0, 0),$$

so

$$\sum_{i=1}^{m} \alpha_i v_i = 0 \quad \text{in } V, \qquad \sum_{j=1}^{n} \beta_j w_j = 0 \quad \text{in } W.$$

Because $\{v_1, \dots, v_m\}$ is a basis of V, it is linearly independent, so

$$\alpha_1 = \cdots = \alpha_m = 0.$$

Similarly, $\{w_1, \dots, w_n\}$ is linearly independent, so

$$\beta_1 = \cdots = \beta_n = 0.$$

Hence the only linear combination of vectors in $\mathcal{B}$ that gives $(0, 0)$ is the trivial one, so $\mathcal{B}$ is linearly independent.

Since $\mathcal{B}$ both spans Z and is linearly independent, it is a basis of Z. The set $\mathcal{B}$ contains $m + n$ vectors, so

$$\dim Z = m + n.$$

□

Xingjian Xu
xingjianxu@ufl.edu
LIT 453
Webpage: https://people.clas.ufl.edu/xingjianxu/mas4105-26springs/
Office Hours: Tuesday, Periods 3 and 5 on LIT453;