Question 1: Label the following statements as true or false.

(a).The zero vector space has no basis.

(b).Every vector space that is generated by a finite set has a basis.

(c). Every vector space has a finite basis.

(d). A vector space cannot have more than one basis.

(e). If a vector space has a finite basis, then the number of vectors in every basis is the same.

(f). The dimension of $\mathbf{P}_n(F)$ is n.

(g). The dimension of $\mathbf{M}_{m\times n}(F)$ is $m+n$.

(h). Suppose that $\mathbf{V}$ is a finite-dimensional vector space, that S_1 is a linearly independent subset of $\mathbf{V}$, and that S_2 is a subset of $\mathbf{V}$ that generates $\mathbf{V}$. Then S_1 cannot contain more vectors than S_2.

(i). If S generates the vector space $\mathbf{V}$, then every vector in $\mathbf{V}$ can be written as a linear combination of vectors in S in only one way.

(j). Every subspace of a finite-dimensional space is finite-dimensional.

(k). If $\mathbf{V}$ is a vector space having dimension n, then $\mathbf{V}$ has exactly one subspace with dimension 0 and exactly one subspace with dimension n.

(l). If $\mathbf{V}$ is a vector space having dimension n, and if S is a subset of $\mathbf{V}$ with n vectors, then S is linearly independent if and only if S spans $\mathbf{V}$

Xingjian's Solution:

(a) False. The zero vector space has the empty set as a basis.

(b) True. Any finite generating set contains a basis.

(c) False. Infinite-dimensional vector spaces exist.

(d) False. A vector space may have many different bases.

(e) True. All bases of a finite-dimensional vector space have the same number of vectors.

(f) False. $\dim \mathbf{P}_n(F) = n+1$.

(g) False. $\dim \mathbf{M}_{m\times n}(F) = mn$.

(h) True. Any linearly independent set has size at most that of any spanning set.

(i) False. Uniqueness holds only for a basis.

(j) True. Every subspace of a finite-dimensional space is finite-dimensional.

(k) True. The only subspaces of dimensions 0 and n are $\{0\}$ and $\mathbf{V}$.

(l) True. A set of n vectors in an n-dimensional space is linearly independent if and only if it spans.

Question 2: Determine which of the following sets are bases for $\mathbb{R}^3$.
(a). $\{(1,0,-1),(2,5,1),(0,-4,3)\}$
Xingjian's Solution: Let

$$v_1 = (1,0,-1), \quad v_2 = (2,5,1), \quad v_3 = (0,-4,3).$$

Method 1 (linear system). Consider

$$av_1 + bv_2 + cv_3 = (0,0,0).$$

This gives

$$a(1,0,-1) + b(2,5,1) + c(0,-4,3) = (0,0,0),$$

so

$$\begin{cases} a + 2b + 0c = 0, \\ 0a + 5b - 4c = 0, \\ -a + b + 3c = 0. \end{cases}$$

From the first equation, $a = -2b$. Substitute into the third:

$$-(-2b) + b + 3c = 0 \implies 2b + b + 3c = 0 \implies 3b + 3c = 0,$$

so $b = -c$. Plug into the second equation:

$$5(-c) - 4c = -9c = 0 \implies c = 0.$$

Hence $b = 0$ and $a = -2b = 0$. Thus the only solution is

$$(a,b,c) = (0,0,0),$$

so $\{v_1, v_2, v_3\}$ is linearly independent.

Xingjian comment: In Method 1, we test whether vectors form a basis by asking a single question: does the equation

$$av_1 + bv_2 + cv_3 = 0$$

have any solution other than $a = b = c = 0$? This question is exactly the definition of linear independence.

To turn this vector equation into something we can compute, we write each vector in coordinates. Placing v_1, v_2, v_3 as columns of a matrix A simply records their coordinates in a compact way. With this notation, the vector equation above becomes the matrix equation

$$A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.$$

Thus, solving the linear combination in Method 1 is the same as solving this homogeneous linear system. But how do we solve it without computing a, b, c?The key observation is that for a square system, the determinant measures whether the columns of the matrix provide enough

independent directions to determine the unknowns uniquely. If $\det(A) \neq 0$, then A is invertible, which means the linear transformation $x \mapsto Ax$ does not collapse any nonzero vector to zero. As a result, the equation $A\mathbf{x} = 0$ forces $\mathbf{x} = 0$, so the system has only the trivial solution.

On the other hand, if $\det(A) = 0$, then A is not invertible. In this case, the columns of A fail to provide independent directions, and the transformation Ax collapses some nonzero vectors to zero. This guarantees the existence of nontrivial solutions to $A\mathbf{x} = 0$, meaning one of the vectors can be written as a linear combination of the others.

Therefore, the determinant allows us to decide linear independence directly, without solving for the unknown coefficients. It serves as a quick test for whether the vectors form a basis, compressing the computations in Method 1 into a single scalar quantity

Method 2 (determinant corollary). Form the matrix with these vectors as columns:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ -1 & 1 & 3 \end{pmatrix}.$$

Its determinant is

$$\det A = 1\begin{vmatrix} 5 & -4 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & -4 \\ -1 & 3 \end{vmatrix} + 0 \cdot (\cdots) = 1(15+4) - 2(0-4) = 19 + 8 = 27 \neq 0.$$

Hence the columns are linearly independent and form a basis for $\mathbb{R}^3$.

Therefore, the set in (a) *is* a basis for $\mathbb{R}^3$.

(b). $\{(2,-4,1),(0,3,-1),(6,0,-1)\}$

Xingjian's Solution: Let $v_1 = (2,-4,1)$, $v_2 = (0,3,-1)$, $v_3 = (6,0,-1)$.

Method 1 (linear system). Consider $av_1 + bv_2 + cv_3 = 0$. That is $a(2,-4,1) + b(0,3,-1) + c(6,0,-1) = (0,0,0)$, so

$$\begin{cases} 2a + 0b + 6c = 0, \\ -4a + 3b + 0c = 0, \\ a - b - c = 0. \end{cases}$$

From the first equation, $2a + 6c = 0$, so $a = -3c$. Substitute into the third:

$$(-3c) - b - c = 0 \implies -4c - b = 0 \implies b = -4c.$$

Check the second equation:

$$-4(-3c) + 3(-4c) = 12c - 12c = 0,$$

so it is satisfied for all c. Thus

$$(a,b,c) = (-3c,-4c,c) = c(-3,-4,1),$$

and taking $c = 1$ gives a nontrivial relation

$$-3v_1 - 4v_2 + v_3 = 0 \quad \iff \quad v_3 = 3v_1 + 4v_2.$$

Hence the set is linearly dependent.

Method 2 (determinant corollary). The matrix of columns is

$$B = \begin{pmatrix} 2 & 0 & 6 \\ -4 & 3 & 0 \\ 1 & -1 & -1 \end{pmatrix}.$$

We compute

$$\det B = 2\begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix} - 0 \cdot (\cdots) + 6\begin{vmatrix} -4 & 3 \\ 1 & -1 \end{vmatrix} = 2(-3) + 6(4-3) = -6 + 6 = 0,$$

so $\det B = 0$. Thus the columns are linearly dependent, consistent with Method 1.

Therefore, the set in (b) is *not* a basis.

(c). $\{(1, 2, -1), (1, 0, 2), (2, 1, 1)\}$

Xingjian's Solution: Let $v_1 = (1, 2, -1), \quad v_2 = (1, 0, 2), \quad v_3 = (2, 1, 1)$.

Method 1 (linear system). Consider $av_1 + bv_2 + cv_3 = 0$, that is

$$a(1, 2, -1) + b(1, 0, 2) + c(2, 1, 1) = (0, 0, 0),$$

so

$$\begin{cases} a + b + 2c = 0, \\ 2a + 0b + c = 0, \\ -a + 2b + c = 0. \end{cases}$$

From the second equation, $2a + c = 0$, so $c = -2a$. Substitute into the first:

$$a + b + 2(-2a) = 0 \implies a + b - 4a = 0 \implies -3a + b = 0,$$

so $b = 3a$. Substitute b and c into the third:

$$-a + 2(3a) + (-2a) = -a + 6a - 2a = 3a.$$

The equation $3a = 0$ implies $a = 0$, then $b = 3a = 0$ and $c = -2a = 0$. Thus the only solution is $(0, 0, 0)$ and the set is linearly independent.

Method 2 (determinant corollary). The column matrix is

$$C = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}.$$

Compute

$$\det C = 1\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} = 1(0-2) - 1(2+1) + 2(4-0) = -2 - 3 + 8 = 3 \neq 0.$$

Thus the columns are linearly independent, so the set is a basis for $\mathbb{R}^3$.

(d). $\{(-1, 3, 1), (2, -4, -3), (-3, 8, 2)\}$

Xingjian's Solution: Let $v_1 = (-1, 3, 1), \quad v_2 = (2, -4, -3), \quad v_3 = (-3, 8, 2)$.

Method 1 (linear system). Consider $av_1 + bv_2 + cv_3 = 0$, or $a(-1, 3, 1) + b(2, -4, -3) + c(-3, 8, 2) = (0, 0, 0)$, so

$$\begin{cases} -a + 2b - 3c = 0, \\ 3a - 4b + 8c = 0, \\ a - 3b + 2c = 0. \end{cases}$$

From the first equation, $-a + 2b - 3c = 0$, so $a = 2b - 3c$. Substitute into the third:

$$(2b - 3c) - 3b + 2c = 0 \implies -b - c = 0 \implies b = -c.$$

Then $a = 2(-c) - 3c = -5c$. Substitute into the second equation:

$$3(-5c) - 4(-c) + 8c = -15c + 4c + 8c = -3c = 0,$$

so $c = 0$. Hence $b = 0$ and $a = 0$, so only the trivial solution exists and the set is linearly independent.

Method 2 (determinant corollary). The matrix of columns is

$$D = \begin{pmatrix} -1 & 2 & -3 \\ 3 & -4 & 8 \\ 1 & -3 & 2 \end{pmatrix}.$$

Compute

$$\det D = -1\begin{vmatrix} -4 & 8 \\ -3 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 8 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & -4 \\ 1 & -3 \end{vmatrix} = -1((-8+24)) - 2(6-8) - 3(-9+4) = -16+4+15 = 3 \neq 0.$$

Hence the set is linearly independent and thus a basis.

(e). $\{(1, -3, -2), (-3, 1, 3), (-2, -10, -2)\}$

Xingjian's Solution: Let $v_1 = (1, -3, -2), \quad v_2 = (-3, 1, 3), \quad v_3 = (-2, -10, -2)$.

Method 1 (linear system). Consider $av_1 + bv_2 + cv_3 = 0$, or $a(1, -3, -2) + b(-3, 1, 3) + c(-2, -10, -2) = (0, 0, 0)$, so

$$\begin{cases} a - 3b - 2c = 0, \\ -3a + b - 10c = 0, \\ -2a + 3b - 2c = 0. \end{cases}$$

From the first equation, $a = 3b + 2c$. Substitute into the third:

$$-2(3b + 2c) + 3b - 2c = 0 \implies -6b - 4c + 3b - 2c = 0 \implies -3b - 6c = 0,$$

so $b = -2c$. Then $a = 3(-2c) + 2c = -6c + 2c = -4c$. Check the second equation:

$$-3(-4c) + (-2c) - 10c = 12c - 2c - 10c = 0,$$

so it holds for all c. Thus

$$(a, b, c) = (-4c, -2c, c) = c(-4, -2, 1),$$

and taking $c = 1$ yields

$$-4v_1 - 2v_2 + v_3 = 0 \quad \Longleftrightarrow \quad v_3 = 4v_1 + 2v_2.$$

Hence the vectors are linearly dependent.

Method 2 (determinant corollary). The column matrix is

$$E = \begin{pmatrix} 1 & -3 & -2 \\ -3 & 1 & -10 \\ -2 & 3 & -2 \end{pmatrix}.$$

Compute

$$\det E = 1\begin{vmatrix} 1 & -10 \\ 3 & -2 \end{vmatrix} + 3\begin{vmatrix} -3 & -10 \\ -2 & -2 \end{vmatrix} - 2\begin{vmatrix} -3 & 1 \\ -2 & 3 \end{vmatrix} = 1(-2+30)+3(6-20)-2(-9+2) = 28-42+14 = 0,$$

which shows the columns are linearly dependent, consistent with Method 1.

Question 3: Determine which of the following sets are bases for $\mathbf{P}_2(\mathbb{R})$.

Recall that $\mathbf{P}_2(\mathbb{R})$ has dimension 3 with standard basis $\{1, x, x^2\}$. A set of three polynomials is a basis if and only if it is linearly independent.

(a). $\{-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2\}$

Xingjian's Solution: Let $p_1 = -1 - x + 2x^2, \quad p_2 = 2 + x - 2x^2, \quad p_3 = 1 - 2x + 4x^2$.

Method 1 (linear combination). Assume $ap_1(x)+bp_2(x)+cp_3(x) = 0$ for all x. Expanding,

$$a(-1 - x + 2x^2) + b(2 + x - 2x^2) + c(1 - 2x + 4x^2) = 0.$$

So we get the system

$$\begin{cases} -a + 2b + c = 0, \\ -a + b - 2c = 0, \\ 2a - 2b + 4c = 0. \end{cases}$$

From the first equation, $-a + 2b + c = 0 \implies a = 2b + c$. Substitute into the second:

$$-(2b + c) + b - 2c = 0 \implies -2b - c + b - 2c = 0 \implies -b - 3c = 0 \implies b = -3c.$$

Then $a = 2b + c = 2(-3c) + c = -6c + c = -5c$.

Check the third equation:

$$2a - 2b + 4c = 2(-5c) - 2(-3c) + 4c = -10c + 6c + 4c = 0,$$

so it is automatically satisfied.

Thus

$$(a, b, c) = (-5c,\ -3c,\ c), \quad c \in \mathbb{R}.$$

Taking, for example, $c = 1$, we obtain a nontrivial relation

$$-5p_1 - 3p_2 + p_3 = 0 \quad \Longleftrightarrow \quad p_3 = 5p_1 + 3p_2.$$

Therefore the polynomials are linearly dependent, so the set is *not* a basis.

Method 2 (determinant corollary). Write each polynomial in the basis $\{1, x, x^2\}$:

$$p_1 \sim (-1, -1, 2), \quad p_2 \sim (2, 1, -2), \quad p_3 \sim (1, -2, 4).$$

So the coefficient matrix is

$$A = \begin{pmatrix} -1 & 2 & 1 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}.$$

Compute the determinant by expanding along the first row:

$$\det(A) = -1 \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} - 2 \begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix}.$$

So

$$\det(A) = -1 \cdot 0 - 2 \cdot 0 + 1 \cdot 0 = 0.$$

Hence the columns are linearly dependent, agreeing with Method 1.

(b). $\{1 + 2x + x^2, 3 + x, x + x^2\}$

Xingjian's Solution: Let $p_1(x) = 1 + 2x + x^2$, $p_2(x) = 3 + x$, $p_3(x) = x + x^2$.

Method 1 (linear combination). Assume $ap_1(x) + bp_2(x) + cp_3(x) = 0$. Expand:

$$a(1 + 2x + x^2) + b(3 + x) + c(x + x^2) = 0.$$

Thus

$$\begin{cases} a + 3b = 0, \\ 2a + b + c = 0, \\ a + c = 0. \end{cases}$$

From the third equation, $c = -a$. Substitute into the second:

$$2a + b - a = 0 \implies a + b = 0 \implies b = -a.$$

Now substitute $b = -a$ into the first:

$$a + 3(-a) = a - 3a = -2a = 0 \implies a = 0.$$

Then $b = -a = 0$ and $c = -a = 0$.

So the only solution is $a = b = c = 0$, hence the set is linearly independent.

Method 2 (determinant corollary). In coordinates:

$$p_1 \sim (1, 2, 1), \quad p_2 \sim (3, 1, 0), \quad p_3 \sim (0, 1, 1).$$

So

$$B = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Compute

$$\det(B) = 1\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} - 3\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix} + 0\cdot(\cdots).$$

We have

$$\det(B) = 1\cdot 1 - 3\cdot 1 = -2 \neq 0.$$

So the set is a basis for $\mathbf{P}_2(\mathbb{R})$.

(c). $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$

Xingjian's Solution: Let $p_1(x) = 1-2x-2x^2$, $p_2(x) = -2+3x-x^2$, $p_3(x) = 1-x+6x^2$.

Method 1 (linear combination). Assume $ap_1(x)+bp_2(x)+cp_3(x) = 0$. Expand:

$$a(1-2x-2x^2) + b(-2+3x-x^2) + c(1-x+6x^2) = 0.$$

So

$$\begin{cases} a-2b+c=0,\\ -2a+3b-c=0,\\ -2a-b+6c=0. \end{cases}$$

From the first equation, $c = -a+2b$. Substitute into the second:

$$-2a+3b-(-a+2b) = 0 \implies -2a+3b+a-2b=0 \implies -a+b=0 \implies b=a.$$

Then $c = -a+2a = a$.

Substitute into the third equation:

$$-2a-b+6c = -2a-a+6a = 3a = 0 \implies a = 0.$$

Thus $b = a = 0$ and $c = a = 0$, so only the trivial solution exists and the set is linearly independent.

Method 2 (determinant colloary). Coordinates: $p_1 \sim (1,-2,-2)$, $p_2 \sim (-2,3,-1)$, $p_3 \sim (1,-1,6)$. So

$$C = \begin{pmatrix} 1 & -2 & 1\\ -2 & 3 & -1\\ -2 & -1 & 6 \end{pmatrix}.$$

Compute

$$\det(C) = 1\begin{vmatrix}3 & -1\\ -1 & 6\end{vmatrix} - (-2)\begin{vmatrix}-2 & -1\\ -2 & 6\end{vmatrix} + 1\begin{vmatrix}-2 & 3\\ -2 & -1\end{vmatrix}.$$

Therefore

$$\det(C) = 1\cdot 17 - (-2)\cdot(-14) + 1\cdot 8 = 17-28+8 = -3 \neq 0.$$

Thus the polynomials form a basis.

(d). $\{-1+2x+4x^2, 3-4x-10x^2, -2-5x-6x^2\}$

Xingjian's Solution: Let $p_1(x) = -1+2x+4x^2$, $p_2(x) = 3-4x-10x^2$, $p_3(x) = -2-5x-6x^2$.

Method 1 (linear combination). Assume $ap_1(x) + bp_2(x) + cp_3(x) = 0$. Expand:

$$a(-1 + 2x + 4x^2) + b(3 - 4x - 10x^2) + c(-2 - 5x - 6x^2) = 0.$$

So

$$\begin{cases} -a + 3b - 2c = 0, \\ 2a - 4b - 5c = 0, \\ 4a - 10b - 6c = 0. \end{cases}$$

From the first equation,

$$-a + 3b - 2c = 0 \implies a = 3b - 2c.$$

Substitute $a = 3b - 2c$ into the second:

$$2(3b - 2c) - 4b - 5c = 0 \implies 6b - 4c - 4b - 5c = 0 \implies 2b - 9c = 0 \implies b = \frac{9}{2}c.$$

Now substitute $a = 3b - 2c$ into the third equation:

$$4(3b - 2c) - 10b - 6c = 0 \implies 12b - 8c - 10b - 6c = 0 \implies 2b - 14c = 0 \implies b = 7c.$$

Comparing the two expressions for b, we get

$$\frac{9}{2}c = 7c \implies \left(\frac{9}{2} - 7\right)c = 0 \implies -\frac{5}{2}c = 0 \implies c = 0.$$

Then $b = 0$ and $a = 3b - 2c = 0$. So the only solution is $a = b = c = 0$, hence the set is linearly independent.

(A slightly cleaner route is: after substituting $a = 3b - 2c$ into the second and third equations, we get two linear equations $2b - 9c = 0$ and $2b - 14c = 0$. Subtracting them gives $5c = 0 \Rightarrow c = 0$, and then $b = 0$ and $a = 0$.)

Method 2 (determinant colloary). Coordinates: $p_1 \sim (-1, 2, 4)$, $p_2 \sim (3, -4, -10)$, $p_3 \sim (-2, -5, -6)$. So

$$D = \begin{pmatrix} -1 & 3 & -2 \\ 2 & -4 & -5 \\ 4 & -10 & -6 \end{pmatrix}.$$

Compute

$$\det(D) = -1\begin{vmatrix} -4 & -5 \\ -10 & -6 \end{vmatrix} - 3\begin{vmatrix} 2 & -5 \\ 4 & -6 \end{vmatrix} + (-2)\begin{vmatrix} 2 & -4 \\ 4 & -10 \end{vmatrix}.$$

$$\det(D) = -1(-26) - 3(8) + (-2)(-4) = 26 - 24 + 8 = 10 \neq 0.$$

So the set is a basis for $\mathbf{P}_2(\mathbb{R})$.

(e). $\{1 + 2x - x^2, 4 - 2x + x^2, -1 + 18x - 9x^2\}$

Xingjian's Solution: Let

$$p_1(x) = 1 + 2x - x^2, \quad p_2(x) = 4 - 2x + x^2, \quad p_3(x) = -1 + 18x - 9x^2.$$

Method 1 (linear combination). Assume $ap_1(x) + bp_2(x) + cp_3(x) = 0$. Expand:

$$a(1 + 2x - x^2) + b(4 - 2x + x^2) + c(-1 + 18x - 9x^2) = 0.$$

So

$$\begin{cases} a + 4b - c = 0, \\ 2a - 2b + 18c = 0, \\ -a + b - 9c = 0. \end{cases}$$

From the first equation, $c = a + 4b$. Substitute into the third:

$$-a + b - 9(a + 4b) = 0 \implies -a + b - 9a - 36b = 0 \implies -10a - 35b = 0 \implies 2a + 7b = 0.$$

Now substitute $c = a + 4b$ into the second:

$$2a - 2b + 18(a + 4b) = 0 \implies 2a - 2b + 18a + 72b = 0 \implies 20a + 70b = 0 \implies 2a + 7b = 0.$$

So the second and third equations both reduce to the same condition $2a + 7b = 0$. This gives one constraint on a, b, so we have a one-parameter family of solutions.

For instance, let c be the parameter and solve directly another way: from the third equation $-a + b - 9c = 0$ and the first $a + 4b - c = 0$, we can solve in terms of c. Add the first and third:

$$(-a + b - 9c) + (a + 4b - c) = 0 \implies 5b - 10c = 0 \implies b = 2c.$$

Then from the first,

$$a + 4(2c) - c = 0 \implies a + 8c - c = 0 \implies a + 7c = 0 \implies a = -7c.$$

Thus

$$(a, b, c) = (-7c,\ 2c,\ c), \quad c \in \mathbb{R}.$$

Taking $c = 1$ gives a nontrivial relation

$$-7p_1 + 2p_2 + p_3 = 0 \quad \iff \quad p_3 = 7p_1 - 2p_2.$$

So the set is linearly dependent and not a basis.

Method 2 (determinant colloary). Coordinates: $p_1 \sim (1, 2, -1)$, $p_2 \sim (4, -2, 1)$, $p_3 \sim (-1, 18, -9)$. So

$$E = \begin{pmatrix} 1 & 4 & -1 \\ 2 & -2 & 18 \\ -1 & 1 & -9 \end{pmatrix}.$$

Compute

$$\det(E) = 1 \begin{vmatrix} -2 & 18 \\ 1 & -9 \end{vmatrix} - 4 \begin{vmatrix} 2 & 18 \\ -1 & -9 \end{vmatrix} + (-1) \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix}.$$

$$\det(E) = 1 \cdot 0 - 4 \cdot 0 + (-1) \cdot 0 = 0.$$

So the columns are linearly dependent, agreeing with Method 1.

Question 4: Do the polynomials $x^3 - 2x^2 + 1, 4x^2 - x + 3$, and $3x - 2$ generate $\mathbf{P}_3(\mathbb{R})$? Justify your answer.

Xingjian's Solution:

Proof. Let the polynomials $p_1(x) = x^3 - 2x^2 + 1, \quad p_2(x) = 4x^2 - x + 3, \quad p_3(x) = 3x - 2$. The vector space $\mathbf{P}_3(\mathbb{R})$ consists of all real polynomials of degree at most 3 and has dimension

$$\dim \mathbf{P}_3(\mathbb{R}) = 4,$$

with standard basis $\{1, x, x^2, x^3\}$.

Any spanning set (generating set) of a finite-dimensional vector space must have at least as many elements as the dimension of the space. Since we are given only 3 polynomials, the subspace they span has dimension at most 3, and therefore cannot equal all of $\mathbf{P}_3(\mathbb{R})$, which has dimension 4.

Hence $\{p_1, p_2, p_3\}$ does not generate $\mathbf{P}_3(\mathbb{R})$. □

Question 6: Give three different bases for $\mathbf{F}^2$ and for $M_{2\times 2}(F)$.
Xingjian's Solution:

Proof. **For $\mathbf{F}^2$:** Since $\dim \mathbf{F}^2 = 2$, any two linearly independent vectors form a basis. For example, the following are three different bases:

- Standard basis:
$$B_1 = \{(1,0), (0,1)\}.$$
- Rotated basis:
$$B_2 = \{(1,1), (1,-1)\}.$$
- Another independent pair:
$$B_3 = \{(2,3), (1,4)\}.$$

In each case, the two vectors are not multiples of each other, so they are linearly independent and span $\mathbf{F}^2$; hence each set is a basis.

For $M_{2\times 2}(F)$: This space has dimension 4, so any set of four linearly independent 2×2 matrices forms a basis. Here are three different bases.

- Standard matrix units:
$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
so
$$B_1 = \{E_{11}, E_{12}, E_{21}, E_{22}\}.$$
- A basis obtained by replacing one matrix with a linear combination:
$$B_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$
Here the second matrix is $E_{11} + E_{12}$, and it is easy to check these four matrices are still linearly independent.

- Another distinct basis:

$$B_3 = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Each of these sets has four linearly independent matrices, so each is a basis of $M_{2\times 2}(F)$. □

Question 11: Let u and v be distinct vectors of a vector space $\mathbf{V}$. Show that if $\{u, v\}$ is a basis for $\mathbf{V}$ and a and b are nonzero scalars, then both $\{u + v, au\}$ and $\{au, bv\}$ are also bases for $\mathbf{V}$.

Xingjian's Solution:

Proof. Let u and v be distinct vectors in $\mathbf{V}$ such that $\{u, v\}$ is a basis. Then u and v are linearly independent and span $\mathbf{V}$, so $\dim \mathbf{V} = 2$.

(1) Show that $\{u + v, au\}$ is a basis.

Since $\mathbf{V}$ is 2-dimensional, it suffices to show that $u + v$ and au are linearly independent.

Suppose

$$\alpha(u + v) + \beta(au) = 0$$

for some scalars α, β. Then

$$\alpha u + \alpha v + a\beta u = (\alpha + a\beta)u + \alpha v = 0.$$

Because $\{u, v\}$ is linearly independent, both coefficients must be zero:

$$\alpha = 0, \qquad \alpha + a\beta = 0.$$

From $\alpha = 0$ we get $a\beta = 0$. Since $a \neq 0$, it follows that $\beta = 0$.

Thus the only solution is $\alpha = \beta = 0$, so $\{u + v, au\}$ is linearly independent. Two independent vectors in a 2-dimensional space form a basis, so $\{u + v, au\}$ is a basis of $\mathbf{V}$.

(2) Show that $\{au, bv\}$ is a basis.

Again, it suffices to show linear independence. Suppose

$$\alpha(au) + \beta(bv) = 0.$$

Then

$$(\alpha a)u + (\beta b)v = 0.$$

Since $\{u, v\}$ is linearly independent, we must have

$$\alpha a = 0, \qquad \beta b = 0.$$

Because a and b are nonzero scalars, this implies $\alpha = 0$ and $\beta = 0$. Hence $\{au, bv\}$ is linearly independent, and with two vectors in a 2-dimensional space, it is a basis.

Therefore both $\{u + v, au\}$ and $\{au, bv\}$ are bases for $\mathbf{V}$. □

Question 13: The set of solutions to the system of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}.$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

Xingjian's Solution:

Proof. Subtract the first equation from the second:

$$(2x_1 - 3x_2 + x_3) - (x_1 - 2x_2 + x_3) = 0 \implies x_1 - x_2 = 0 \implies x_1 = x_2.$$

Substitute $x_1 = x_2$ into the first equation:

$$x_2 - 2x_2 + x_3 = 0 \implies -x_2 + x_3 = 0 \implies x_3 = x_2.$$

Let $t = x_2$. Then

$$x_1 = t, \quad x_2 = t, \quad x_3 = t,$$

so every solution has the form

$$(x_1, x_2, x_3) = (t, t, t) = t(1, 1, 1), \quad t \in \mathbb{R}.$$

Thus the solution set is the line spanned by $(1, 1, 1)$, which is a subspace of $\mathbb{R}^3$. A basis for this subspace is

$$\{(1, 1, 1)\}.$$

□

Question 16: The set of all upper triangular $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

Xingjian's Solution:

Proof. Let $W \subset M_{n\times n}(F)$ be the set of all upper triangular matrices. A matrix $A = (a_{ij})$ is in W if and only if $a_{ij} = 0$ whenever $i > j$.

Thus, the entries on and above the main diagonal (i.e. with $i \leq j$) may be arbitrary, while the entries below the main diagonal ($i > j$) are all zero. A natural basis for W is given by the standard matrix units

$$E_{ij}, \qquad 1 \leq i \leq j \leq n,$$

where E_{ij} has a 1 in the (i, j)-entry and 0 elsewhere. Any upper triangular matrix can be written uniquely as a linear combination of these E_{ij}.

Therefore,

$$\mathcal{B} = \{E_{ij} : 1 \leq i \leq j \leq n\}$$

is a basis for W.

To find the dimension, we count the number of pairs (i, j) with $1 \leq i \leq j \leq n$. For each fixed j, the index i can take the values $1, 2, \ldots, j$, giving j options. Hence

$$\dim W = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

□

Question 26: For a fixed $a \in \mathbb{R}$, determine the dimension of the subspace of $\mathbf{P}_n(\mathbb{R})$ defined by $\{f \in \mathbf{P}_n(\mathbb{R}) : f(a) = 0\}$.

Xingjian's Solution: We consider the subspace

$$U = \{f \in \mathbf{P}_n(\mathbb{R}) : f(a) = 0\},$$

where $a \in \mathbb{R}$ is fixed.

Proof. Recall that $\mathbf{P}_n(\mathbb{R})$ consists of all real polynomials of degree at most n and has dimension $n+1$, with standard basis

$$\{1, x, x^2, \ldots, x^n\}.$$

First, note that if $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$. Thus every polynomial $f \in U$ can be written in the form

$$f(x) = (x-a)g(x),$$

where $g(x)$ is a polynomial of degree at most $n-1$.

Conversely, for any polynomial $g \in \mathbf{P}_{n-1}(\mathbb{R})$ of degree at most $n-1$, the polynomial $f(x) = (x-a)g(x)$ satisfies $f(a) = 0$, so $f \in U$. Hence every element of U arises in this way.

Now consider the set

$$\mathcal{B} = \{(x-a), (x-a)x, (x-a)x^2, \ldots, (x-a)x^{n-1}\}.$$

Each polynomial in $\mathcal{B}$ belongs to U, and there are exactly n of them.

(I).$a = 0$,

$$\mathcal{B} = \{x, x^2, x^3, \ldots, x^n\}.$$

Then it is a basis; so $\dim U = n$ as a trivial case.

(II). if $a \neq 0$, We claim that $\mathcal{B}$ is a basis for U.

Spanning. Let $f \in U$. Then $f(x) = (x-a)g(x)$ for some $g(x) \in \mathbf{P}_{n-1}(\mathbb{R})$. Since $\{1, x, x^2, \ldots, x^{n-1}\}$ is a basis for $\mathbf{P}_{n-1}(\mathbb{R})$, we may write

$$g(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}.$$

Thus

$$f(x) = (x-a)(c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}),$$

which is a linear combination of the elements of $\mathcal{B}$. Hence $\mathcal{B}$ spans U.

Linear independence. Suppose

$$c_0(x-a) + c_1(x-a)x + \cdots + c_{n-1}(x-a)x^{n-1} = 0.$$

Factoring out $(x-a)$ gives

$$(x-a)\big(c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}\big) = 0.$$

Since $(x-a)$ is not the zero polynomial, the factor in parentheses must be the zero polynomial. Thus

$$c_0 = c_1 = \cdots = c_{n-1} = 0.$$

Hence the set $\mathcal{B}$ is linearly independent.

Therefore $\mathcal{B}$ is a basis for U, and since it has n elements, we conclude that

$$\dim U = n.$$

□

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