Xingjian comment: In the following, I have already included not only optional problem from Section 1.5, but also some interesting proof-driven questions in Section 1.4.

Question 11 from Section 1.4: Prove that $\text{Span}(\{x\}) = \{ax : a \in F\}$ for any vector x in a vector space. Interpret this result geometrically in $\mathbb{R}^3$.

Xingjian's Solution:

Proof. We prove the equality by showing both inclusions.

[$\text{Span}(\{x\}) \subseteq \{ax : a \in F\}$]: By definition, any vector $v \in \text{Span}(\{x\})$ is a finite linear combination of vectors from the set $\{x\}$. Since $\{x\}$ contains only one vector, there exists a scalar $a \in F$ such that $v = ax$. Hence $v \in \{ax : a \in F\}$, and therefore

$$\text{Span}(\{x\}) \subseteq \{ax : a \in F\}.$$

[$\{ax : a \in F\} \subseteq \text{Span}(\{x\})$]: Let $v \in \{ax : a \in F\}$. Then $v = ax$ for some $a \in F$. Since v is a scalar multiple of x, it is a linear combination of vectors from $\{x\}$, and thus $v \in \text{Span}(\{x\})$. Therefore,

$$\{ax : a \in F\} \subseteq \text{Span}(\{x\}).$$

We conclude that $\text{Span}(\{x\}) = \{ax : a \in F\}$.

Geometric interpretation in $\mathbb{R}^3$. If $x \neq 0$, the set $\{ax : a \in \mathbb{R}\}$ is the line through the origin in the direction of x. If $x = 0$, then $\text{Span}(\{x\}) = \{0\}$, which corresponds to the origin. □

Question 12 from Section 1.4: Show that a subset W of a vector space V is a subspace of V if and only if $\text{Span}(W) = W$.

Xingjian's Solution:

Proof. ($\Rightarrow$) Suppose W is a subspace of V, $\text{Span}(W)$ is the smallest subspace containing W.[You might think a little bit for the reason.] Since W itself is a subspace containing W, we have $\text{Span}(W) \subseteq W$. Since $W \subseteq \text{Span}(W)$ always holds, it follows that $\text{Span}(W) = W$.

($\Leftarrow$) Suppose $\text{Span}(W) = W$. Since the span of any set is a subspace, W must be a subspace of V. □

Question 13 from Section 1.4: Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{Span}(S_1) \subseteq \text{Span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{Span}(S_1) = V$, deduce that $\text{Span}(S_2) = V$.

Xingjian's Solution:

Proof. Let $v \in \text{Span}(S_1)$. Then $v = a_1 s_1 + \cdots + a_n s_n$ for some $s_i \in S_1$ and non-zero scalars a_i in F. Since $S_1 \subseteq S_2$, each $s_i \in S_2$, so $v \in \text{Span}(S_2)$. Hence, $\text{Span}(S_1) \subseteq \text{Span}(S_2)$.

If $\text{Span}(S_1) = V$, then $V = \text{Span}(S_1) \subseteq \text{Span}(S_2) \subseteq V$, which implies $\text{Span}(S_2) = V$. □

Question 14 from Section 1.4: Show that if S_1 and S_2 are arbitrary subsets for a vector space V, then $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2)$.(The sume of two subsets is defined before.)

Xingjian's Solution:

Proof. We prove both inclusions.

[$\mathrm{Span}(S_1 \cup S_2) \subseteq \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$]: Let $v \in \mathrm{Span}(S_1 \cup S_2)$. By definition of span, there exist vectors $w_1, w_2, \ldots, w_m \in S_1 \cup S_2$ and scalars $a_1, a_2, \ldots, a_m$ in F such that

$$v = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m.$$

Define two index sets: $I_1 = \{\, i : w_i \in S_1 \,\}$, $I_2 = \{\, i : w_i \in S_2 \,\}$. (Every index i belongs to at least one of these, since each $w_i \in S_1 \cup S_2$.) Now define $x := \sum_{i \in I_1} a_i w_i$, $y := \sum_{i \in I_2} a_i w_i$. Then we can rewrite v as

$$v = \sum_{i=1}^{m} a_i w_i = \sum_{i \in I_1} a_i w_i + \sum_{i \in I_2} a_i w_i = x + y.$$

Each w_i with $i \in I_1$ lies in S_1, so x is a linear combination of vectors in S_1, and hence $x \in \mathrm{Span}(S_1)$. Similarly, y is a linear combination of vectors in S_2, so $y \in \mathrm{Span}(S_2)$. Thus $v = x + y \in \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$, and we conclude $\mathrm{Span}(S_1 \cup S_2) \subseteq \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$.

[$\mathrm{Span}(S_1) + \mathrm{Span}(S_2) \subseteq \mathrm{Span}(S_1 \cup S_2)$.]: Let $v \in \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$. Then there exist $x \in \mathrm{Span}(S_1), y \in \mathrm{Span}(S_2)$ such that $v = x + y$.

Since $x \in \mathrm{Span}(S_1)$, we can write $x = \sum_{i=1}^{p} \alpha_i s_i$ for some $s_1, \ldots, s_p \in S_1$ and $\alpha_1, \ldots, \alpha_p \in F$. Similarly, since $y \in \mathrm{Span}(S_2)$, we can write $y = \sum_{j=1}^{q} \beta_j t_j$ for some $t_1, \ldots, t_q \in S_2$ and $\beta_1, \ldots, \beta_q \in F$. Therefore,

$$v = x + y = \sum_{i=1}^{p} \alpha_i s_i + \sum_{j=1}^{q} \beta_j t_j.$$

Each s_i and t_j belongs to $S_1 \cup S_2$, so v is a linear combination of vectors from $S_1 \cup S_2$. Hence $v \in \mathrm{Span}(S_1 \cup S_2)$. This shows $\mathrm{Span}(S_1) + \mathrm{Span}(S_2) \subseteq \mathrm{Span}(S_1 \cup S_2)$.

Combining both inclusions, we obtain $\mathrm{Span}(S_1 \cup S_2) = \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$. □

Question 15 from Section 1.4: Let S_1 and S_2 be subsets of a vector space V. Prove that $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$. Give an example in which $\mathrm{Span}(S_1 \cap S_2)$ and $\mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$ are equal and one in which they are unequal.

Xingjian's Solution:

Proof. If $v \in \mathrm{Span}(S_1 \cap S_2)$, then v is a linear combination of vectors belonging to both S_1 and S_2. Hence $v \in \mathrm{Span}(S_1)$ and $v \in \mathrm{Span}(S_2)$, so

$$v \in \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2).$$

□

Example (equal): Let $S_1 = S_2 = \{(1,0)\} \subset \mathbb{R}^2$.

Example (unequal): Let $S_1 = \{(1,0),(0,1)\}$ and $S_2 = \{(1,0),(1,1)\}$ in $\mathbb{R}^2$.

Question 9 from Section 1.5: Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Xingjian's Solution:

Proof. We prove the statement in both directions.

($\Rightarrow$) Suppose $\{u, v\}$ is linearly dependent. We show that one of u or v is a scalar multiple of the other: Since $\{u, v\}$ is linearly dependent, there exist scalars $a, b \in F$, not both zero, such that

$$au + bv = 0. \tag{1}$$

There are two cases to consider.

Case 1: $a \neq 0$. Then we can solve equation (1) for u. $au = -bv \quad \Rightarrow \quad u = -\frac{b}{a}v$.Thus u is a scalar multiple of v (with scalar $-\frac{b}{a}$).

Case 2: $a = 0$. Because a and b are not both zero, it follows that $b \neq 0$. Then equation (1) becomes $0 \cdot u + bv = 0 \quad \Longrightarrow \quad bv = 0$. Since $b \neq 0$ and we are in a vector space over a field, the only way $bv = 0$ can hold is if $v = 0$. In that case, v is the zero vector, and we can write $v = 0 \cdot u$, so v is a scalar multiple of u (with scalar 0).

In either case, we conclude that one of the vectors is a scalar multiple of the other.

($\Leftarrow$) Conversely, suppose one of u or v is a scalar multiple of the other. We show that $\{u, v\}$ is linearly dependent: There are again two cases.

Case 1: u is a scalar multiple of v. Then there exists a scalar $\lambda \in F$ such that $u = \lambda v$. Rearranging this equation, we get $u - \lambda v = 0$. This is a linear combination of u and v: $1 \cdot u + (-\lambda)v = 0$. The coefficients here are 1 and $-\lambda$. At least one of these coefficients is nonzero (in fact, $1 \neq 0$), so this is a nontrivial linear relation between u and v. This means $\{u, v\}$ is linearly dependent.

Case 2: v is a scalar multiple of u. Then there exists a scalar $\mu \in F$ such that $v = \mu u$. Then, $(-\mu)u + 1 \cdot v = 0$. Again, the coefficients $-\mu$ and 1 are not both zero (indeed $1 \neq 0$), so we have a nontrivial linear combination of u and v equal to zero. Therefore, $\{u, v\}$ is linearly dependent. □

Question 14 from Section 1.5: Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \cdots, u_n$ in S such that v is a linear combination of $u_1, u_2, \cdots, u_n$.

Xingjian's Solution:

Proof. We prove two directions.

($\Rightarrow$) Suppose S is linearly dependent. We show that either $S = \{0\}$ or there exist distinct $v, u_1, \ldots, u_n \in S$ with v a linear combination of $u_1, \ldots, u_n$.

Since S is linearly dependent, by definition there exist vectors $w_1, w_2, \ldots, w_k \in S$ and not all zero scalars $a_1, a_2, \ldots, a_k \in F$, such that

$$a_1 w_1 + a_2 w_2 + \cdots + a_k w_k = 0. \tag{2}$$

Among all such nontrivial relations, choose one with the smallest possible number k of vectors. Thus, we assume the relation (2) is a *minimal* nontrivial linear dependence (no nontrivial relation among fewer vectors in S exists).

Since not all of $a_1, \ldots, a_k$ are zero, at least one of them is nonzero. Without loss of generality, suppose $a_1 \neq 0$. We can then solve equation (2) for w_1:

$$a_1 w_1 = -(a_2 w_2 + \cdots + a_k w_k),$$

so

$$w_1 = -\frac{1}{a_1}(a_2 w_2 + \cdots + a_k w_k) = \left(-\frac{a_2}{a_1}\right) w_2 + \cdots + \left(-\frac{a_k}{a_1}\right) w_k.$$

Thus w_1 is a linear combination of $w_2, \ldots, w_k$, and these vectors are all in S. Moreover, the vectors $w_1, w_2, \ldots, w_k$ are distinct (if two were equal, we could combine them and obtain a dependence involving fewer vectors, contradicting minimality).

If $S = \{0\}$, then the first alternative in the statement holds. If $S \neq \{0\}$, then the above reasoning shows that there exist distinct vectors $v := w_1, \quad u_1 := w_2, \ldots, u_n := w_k$ in S such that v is a linear combination of $u_1, \ldots, u_n$. This is exactly the second alternative in the statement.

($\Leftarrow$) Conversely, suppose $S = \{0\}$ or there exist distinct $v, u_1, \ldots, u_n$ in S with v a linear combination of $u_1, \ldots, u_n$. We show that S is linearly dependent.

Case 1: $S = \{0\}$. Then the single vector 0 belongs to S, and we have $1 \cdot 0 = 0$, which is a nontrivial linear combination (the scalar 1 is not zero) that gives the zero vector. Hence S is linearly dependent by definition.

Case 2: $S \neq \{0\}$ and there exist distinct vectors $v, u_1, \ldots, u_n \in S$ such that $v = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$ for some scalars $c_1, \ldots, c_n \in F$.

Move all terms to one side to obtain a linear combination that equals zero:

$$1 \cdot v + (-c_1) u_1 + \cdots + (-c_n) u_n = 0.$$

The coefficients in this combination are $1, -c_1, -c_2, \ldots, -c_n$. At least one of these coefficients is nonzero, namely the coefficient of v, which is 1. Therefore, we have found a nontrivial linear combination of vectors in S equal to 0. By definition, this means S is linearly dependent.

Combining both directions, we finished the proof. □

Question 19 from Section 1.5: Prove that if $\{A_1, A_2, \cdots, A_k\}$ is a linearly independent subset of $M_{n\times n}(F)$, then $\{A_1^t, A_2^t, \cdots, A_k^t\}$ is also linearly independent.

Xingjian's Solution:

Proof. We will prove this by contradiction. Assume that $\{A_1^t, A_2^t, \ldots, A_k^t\}$ is not linearly independent. Then this set is linearly dependent. By the definition of linear dependence, there exist not all zero scalars $c_1, c_2, \ldots, c_k \in F$, such that

$$c_1 A_1^t + c_2 A_2^t + \cdots + c_k A_k^t = 0_{n\times n},$$

where $0_{n\times n}$ is the $n \times n$ zero matrix.

Now take the transpose of both sides of this matrix equation. Using the facts that $(B + C)^t = B^t + C^t$ for matrices B, C and $(\lambda B)^t = \lambda B^t$ for any scalar λ, and also that $(A_i^t)^t = A_i$, we obtain

$$\left(c_1 A_1^t + \cdots + c_k A_k^t\right)^t = c_1 (A_1^t)^t + \cdots + c_k (A_k^t)^t = c_1 A_1 + \cdots + c_k A_k.$$

The transpose of the zero matrix is again the zero matrix, so the equation becomes

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k = 0_{n\times n}.$$

Thus we have found scalars $c_1, \ldots, c_k$, not all zero, such that a linear combination of $A_1, \ldots, A_k$ equals the zero matrix. By the definition of linear dependence, this means that the set $\{A_1, A_2, \ldots, A_k\}$ is linearly dependent.

However, this contradicts the hypothesis of the problem, which states that $\{A_1, A_2, \ldots, A_k\}$ is linearly independent. It is contradiction. □

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