Xingjian comment: In the following, I have a personal taste for using proof by contradiction for those questions about linear independent and linear dependent. You also can try direct proof to convince me if you want.

Question 1: Label the following statement as true or false.

(a). If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

(b). Any set containing the zero vector is linearly dependent.

(c). The empty set is linearly dependent.

(d). Subsets of linearly dependent sets are linearly dependent.

(e). Subsets of linearly independent sets are linearly dependent.

(f). If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \cdots, x_n$ are linearly independent, then all the scalars a_i are zero.

Xingjian's Solution:

(a). **False.** If S is linearly dependent, then there exists *at least one* vector in S that is a linear combination of the others, not necessarily every vector. For example, in $\mathbb{R}^2$, the set

$$S = \{(1,0), (0,1), (0,0)\}$$

is linearly dependent because of the zero vector, but $(1,0)$ is not a linear combination of $(0,1)$ and $(0,0)$.

(b). **True.** If $0 \in S$, then taking the linear combination

$$1 \cdot 0 + 0 \cdot v_2 + \cdots + 0 \cdot v_n = 0$$

gives a nontrivial relation among the vectors in S, so S is linearly dependent.

(c). **False.** By definition, the empty set is linearly *independent*: there is no nontrivial linear combination of zero vectors.

(d). **False.** A subset of a linearly dependent set may be linearly independent. For example, as above, $S = \{(1,0), (0,1), (0,0)\}$ is linearly dependent, but the subset $\{(1,0), (0,1)\}$ is linearly independent.

(e). **False.** In fact, every subset of a linearly independent set is also linearly independent (this is a standard theorem). So the statement as written is false.

(f). **True.** This is exactly the definition of linear independence: the only linear combination

$$a_1x_1 + \cdots + a_nx_n = 0$$

is the trivial combination $a_1 = \cdots = a_n = 0$.

Question 3: In $\mathbf{M}_{2\times 3}(F)$, prove that the following set is linearly dependent.

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Xingjian's Solution:

Proof. Let

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

We prove the set $\{A_1, A_2, A_3, A_4, A_5\}$ is linearly dependent by contradiction.

Assume that these five matrices are linearly independent. Then the only way a linear combination

$$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 + c_5A_5 = 0$$

can hold is if all coefficients $c_1, \ldots, c_5$ are zero.

Now compute $A_1 + A_2 + A_3$ and $A_4 + A_5$:

$$A_1 + A_2 + A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and

$$A_4 + A_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus

$$A_1 + A_2 + A_3 = A_4 + A_5.$$

Rewriting this, we get

$$(-1)A_1 + (-1)A_2 + (-1)A_3 + 1 \cdot A_4 + 1 \cdot A_5 = 0.$$

This is a nontrivial linear relation among $A_1, \ldots, A_5$, since the coefficients $(-1, -1, -1, 1, 1)$ are not all zero. This contradicts our assumption that the matrices were linearly independent.

Therefore, our assumption must be false, and the set $\{A_1, A_2, A_3, A_4, A_5\}$ is linearly dependent. □

Question 4: In $\mathbf{F}^n$, let e_j denote the vector whose j-th coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \cdots, e_n\}$ is linearly independent.

Xingjian's Solution:

Proof. We prove this by contradiction.

Assume that the set $\{e_1, \ldots, e_n\}$ is linearly *dependent*. Then there exist scalars $a_1, \ldots, a_n \in F$, not all zero, such that

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = 0.$$

Write out the left-hand side as a vector. Since e_j has 1 in the j-th position and 0 elsewhere, we have

$$a_1e_1 + \cdots + a_ne_n = (a_1, a_2, \ldots, a_n).$$

Thus the equation becomes

$$(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0).$$

So each coordinate must be zero:

$$a_1 = a_2 = \cdots = a_n = 0.$$

But this contradicts our assumption that not all of the scalars a_i are zero. Hence our assumption that $\{e_1, \ldots, e_n\}$ is linearly dependent is false. Therefore, $\{e_1, \ldots, e_n\}$ is linearly independent. □

Question 5: Show that the set $\{1, x, x^2, \cdots, x^n\}$ is linearly independent.
Xingjian's Solution:

Proof. **Method 1:** Suppose we have a linear combination of these polynomials equal to the zero polynomial:

$$a_0 \cdot 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0,$$

where the right-hand side is the zero polynomial.

The left-hand side is a polynomial of degree at most n, with coefficients $a_0, a_1, \ldots, a_n$. By the definition of equality of polynomials, two polynomials are equal if and only if all their corresponding coefficients agree. The zero polynomial has all coefficients equal to 0, so we must have

$$a_0 = a_1 = \cdots = a_n = 0.$$

Thus the only linear combination of $1, x, \ldots, x^n$ that gives the zero polynomial is the trivial one, so the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

Method 2: We prove this by contradiction.

Suppose the set $\{1, x, x^2, \ldots, x^n\}$ is linearly *dependent*. Then there exist scalars $a_0, a_1, \ldots, a_n \in F$, not all zero, such that

$$a_0 \cdot 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

for all x. In other words, the polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

is the zero polynomial.

Because not all a_i are zero, let k be the largest index such that $a_k \neq 0$. Then the degree of $p(x)$ is k, and the leading term is $a_k x^k$. In particular, $p(x)$ is a nonzero polynomial of degree k.

W also assumed that $p(x)$ is the zero polynomial, but the degree of zero polynomial is -1, and $-1 \neq k > 0$. This is contradiction. □

Question 7: The set of diagonal matrices in $M_{2\times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.
Xingjian's Solution:

Proof. A general diagonal matrix in $M_{2\times 2}(F)$ has the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $a, b \in F$. We can write this as

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then every diagonal matrix is a linear combination of D_1 and D_2, so $\{D_1, D_2\}$ generates the subspace of diagonal matrices.

To see that $\{D_1, D_2\}$ is linearly independent, suppose

$$\alpha D_1 + \beta D_2 = 0.$$

This gives

$$\alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\alpha = 0$ and $\beta = 0$. So $\{D_1, D_2\}$ is linearly independent and generates the subspace of diagonal matrices. □

Question 10: Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

Xingjian's Solution: Be clear for the meaning of multiple. When we say v is a multiple of w, it means that $v = \lambda w$ for some scalar $\lambda \in \mathbb{R}$.

Now consider $v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (1, 1, 0)$. We have $v_3 = v_1 + v_2$, so

$$v_1 + v_2 - v_3 = 0$$

is a nontrivial linear relation, and the vectors are linearly dependent.

However, no one of these is a scalar multiple of another: $(1, 0, 0)$ is not a multiple of $(0, 1, 0)$ or $(1, 1, 0)$, and similarly for the others. □

Question 16: Prove that a set S of vectors is linearly independent if a finite subset of S is linearly independent.

Xingjian's Solution:

Proof. We prove it by contradiction. Suppose that S is linearly *dependent*. By definition, this means that there exists a finite subset $\{v_1, \ldots, v_k\} \subseteq S$ and scalars $a_1, \ldots, a_k$, not all zero, such that

$$a_1 v_1 + \cdots + a_k v_k = 0.$$

But then the finite subset $\{v_1, \ldots, v_k\}$ is linearly dependent.

Therefore, if *every* finite subset of S is linearly independent, S itself cannot be linearly dependent. It is contradiction. □

Question 18: Let S be a set of nonzero polynomials $\mathbf{P}(F)$ such that no two have the same degree. Prove that S is linearly independent.

Xingjian's Solution:

Proof. We prove this by contradiction. Suppose S is not linearly independent. Then there exists a finite subset $\{p_1, p_2, \ldots, p_m\} \subseteq S$ that is linearly dependent. Since no two polynomials in S have the same degree and all are nonzero, we may reorder the polynomial index to get $\{p_1, p_2, \ldots, p_m\} \subseteq S$ with $\deg p_1 < \deg p_2 < \cdots < \deg p_m$.

Since $\{p_1, \ldots, p_m\}$ is linearly dependent, there exist not all zero scalars $a_1, \ldots, a_m \in F$, s.t.

$$a_1 p_1 + a_2 p_2 + \cdots + a_m p_m = 0,$$

where the right-hand side is the zero polynomial.

Consider the polynomial on the left-hand side. Let $\deg p_m = d$, and let the leading coefficient of p_m be $c \neq 0$. Then the term of highest degree in the sum

$$a_1 p_1 + \cdots + a_m p_m$$

is the x^d-term coming from $a_m p_m$, since all other p_i have strictly smaller degree. The coefficient of x^d in the sum is therefore $a_m c$. But the sum equals the zero polynomial, whose coefficient of x^d is 0. Hence

$$a_m c = 0 \quad \Rightarrow \quad a_m = 0,$$

since $c \neq 0$.

Now the relation reduces to

$$a_1 p_1 + \cdots + a_{m-1} p_{m-1} = 0.$$

Applying the same argument to this shorter relation (again looking at the largest degree among $p_1, \ldots, p_{m-1}$), we conclude $a_{m-1} = 0$. Repeating this step, we eventually obtain

$$a_1 = a_2 = \cdots = a_m = 0.$$

This contradicts our assumption that the relation was nontrivial. Hence every finite subset of S is linearly independent, and thus S is linearly independent. □

Question 20: Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent.

Xingjian's Solution:

Proof. We prove this by contradiction. Assume that f and g are linearly dependent. Then there exist scalars $a, b \in \mathbb{R}$, not both zero, such that

$$ae^{rt} + be^{st} = 0 \quad \text{for all } t \in \mathbb{R}. \tag{$*$}$$

Evaluate $(*)$ at $t = 0$. Since $e^{r \cdot 0} = e^{s \cdot 0} = 1$, we obtain

$$a + b = 0. \tag{1}$$

Now differentiate both sides of $(*)$ with respect to t:

$$are^{rt} + bse^{st} = 0 \quad \text{for all } t \in \mathbb{R}.$$

Evaluating at $t = 0$ again gives

$$ar + bs = 0. \tag{2}$$

Thus (a, b) is a (nonzero) solution of the linear system

$$\begin{cases} a + b = 0, \\ ar + bs = 0. \end{cases}$$

In matrix form,

$$\begin{pmatrix} 1 & 1 \\ r & s \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A nontrivial solution $(a, b) \neq (0, 0)$ exists only if the determinant of the coefficient matrix is zero:

$$\det \begin{pmatrix} 1 & 1 \\ r & s \end{pmatrix} = 1 \cdot s - 1 \cdot r = s - r = 0.$$

Hence $s - r = 0$, so $r = s$.

But this contradicts the assumption in the problem that $r \neq s$. □

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