Question 1: Label the following statements as true or false.

(a) The zero vector is a linear combination of any nonempty set of vectors.

(b) The span of $\emptyset$ is $\emptyset$.

(c) If $\mathbf{S}$ is a subset of a vector space $\mathbf{V}$, then span($\mathbf{S}$) equals the intersection of all subspaces of $\mathbf{V}$ that contain $\mathbf{S}$.

(d) In solving a system of linear equations, it is permissible to multiply an equation by any constant.

(e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another.

(f) Every system of linear equations has a solution.

Xingjian's Solution: (a). **True**. Just pick any coefficient to be zero.

(b). **False**. By definition, span($\emptyset$) is the set of all linear combinations of vectors in $\emptyset$. The only linear combination you can form is the "empty" one, which is taken to be the zero vector. Hence

$$\text{span}(\emptyset) = \{\mathbf{0}\} \neq \emptyset.$$

(c) **True.** Every subspaces of which S is a subset contains span(S) and span(S) is a subspace.

(d) **False.** This action will change the solution of one system of linear equations.

(e) **True.**

(f) **False.** Some systems are inconsistent and have no solutions, e.g.

$$\begin{cases} x_1 = 1, \\ x_1 = 2. \end{cases}$$

So not every system has a solution.

Question 2: Solve the following systems of linear equations by the method introduced in this section:

$$(a). \qquad \begin{cases} 2x_1 - 2x_2 - 3x_3 \qquad = 10 \\ 3x_1 - 3x_2 - 2x_3 + 5x_4 = 7 \\ x_1 - x_2 - 2x_3 - x_4 = -3. \end{cases} \tag{1}$$

Xingjian's Solution: First, eliminate x_1 using the third equation. Multiply the third equation by 2 and subtract from the first:

$$2(x_1 - x_2 - 2x_3 - x_4) = 2(-3) \Rightarrow 2x_1 - 2x_2 - 4x_3 - 2x_4 = -6.$$

Now

$$\text{Eq}_1 - 2\text{Eq}_3 : \quad (2x_1 - 2x_2 - 3x_3) - (2x_1 - 2x_2 - 4x_3 - 2x_4) = 10 - (-6),$$

so

$$x_3 + 2x_4 = 16. \tag{I}$$

Next, multiply the third equation by 3 and subtract from the second:

$$3(x_1 - x_2 - 2x_3 - x_4) = 3(-3) \Rightarrow 3x_1 - 3x_2 - 6x_3 - 3x_4 = -9.$$

Then

$$\mathrm{Eq}_2 - 3\mathrm{Eq}_3 : \quad (3x_1 - 3x_2 - 2x_3 + 5x_4) - (3x_1 - 3x_2 - 6x_3 - 3x_4) = 7 - (-9),$$

which gives

$$4x_3 + 8x_4 = 16 \quad \Rightarrow \quad x_3 + 2x_4 = 4. \tag{II}$$

Equations (I) and (II) give

$$x_3 + 2x_4 = 16 \quad \text{and} \quad x_3 + 2x_4 = 4,$$

which is impossible. Hence there is a contradiction and the system has *no solution.*

$$(b). \qquad \begin{cases} 3x_1 - 7x_2 + 4x_3 = 10 \\ x_1 - 2x_2 + x_3 = 3 \\ 2x_1 - x_2 - 2x_3 = 6. \end{cases} \tag{2}$$

Xingjian's Solution:
From the second equation we express x_1 in terms of x_2, x_3:

$$x_1 - 2x_2 + x_3 = 3 \quad \Rightarrow \quad x_1 = 3 + 2x_2 - x_3.$$

Substitute this into the first and third equations.
For the first equation:

$$\begin{aligned} 3x_1 - 7x_2 + 4x_3 &= 10, \\ 3(3 + 2x_2 - x_3) - 7x_2 + 4x_3 &= 10, \\ 9 + 6x_2 - 3x_3 - 7x_2 + 4x_3 &= 10, \\ 9 - x_2 + x_3 &= 10, \end{aligned}$$

so

$$-x_2 + x_3 = 1 \quad \Rightarrow \quad x_3 = 1 + x_2. \tag{A}$$

For the third equation:

$$\begin{aligned} 2x_1 - x_2 - 2x_3 &= 6, \\ 2(3 + 2x_2 - x_3) - x_2 - 2x_3 &= 6, \\ 6 + 4x_2 - 2x_3 - x_2 - 2x_3 &= 6, \\ 6 + 3x_2 - 4x_3 &= 6, \end{aligned}$$

hence

$$3x_2 - 4x_3 = 0. \tag{B}$$

Now substitute (A) into (B):

$$3x_2 - 4(1 + x_2) = 0 \quad \Rightarrow \quad 3x_2 - 4 - 4x_2 = 0 \quad \Rightarrow \quad -x_2 - 4 = 0,$$

so

$$x_2 = -4.$$

Then from (A),

$$x_3 = 1 + x_2 = 1 + (-4) = -3.$$

Finally, use $x_1 = 3 + 2x_2 - x_3$:

$$x_1 = 3 + 2(-4) - (-3) = 3 - 8 + 3 = -2.$$

Therefore, the solution of the system is

$$(x_1, x_2, x_3) = (-2, -4, -3).$$

$$(c). \qquad \begin{cases} x_1 + 2x_2 - x_3 + x_4 = 5 \\ x_1 + 4a_2 \ - 3x_3 - 3x_4 = 6 \\ 2x_1 + 3x_2 - x_3 + 4x_4 = 8. \end{cases} \tag{3}$$

Xingjian's Solution:

We perform the following equation operations:

$$\mathrm{Eq}_2 \leftarrow \mathrm{Eq}_2 - \mathrm{Eq}_1, \qquad \mathrm{Eq}_3 \leftarrow \mathrm{Eq}_3 - 2\mathrm{Eq}_1.$$

For the second equation:

$$\begin{aligned} (x_1 + 4x_2 - 3x_3 - 3x_4) - (x_1 + 2x_2 - x_3 + x_4) &= 6 - 5, \\ (4x_2 - 2x_2) + (-3x_3 + x_3) + (-3x_4 - x_4) &= 1, \\ 2x_2 - 2x_3 - 4x_4 &= 1. \end{aligned}$$

For the third equation:

$$\begin{aligned} (2x_1 + 3x_2 - x_3 + 4x_4) - 2(x_1 + 2x_2 - x_3 + x_4) &= 8 - 10, \\ (2x_1 - 2x_1) + (3x_2 - 4x_2) + (-x_3 + 2x_3) + (4x_4 - 2x_4) &= -2, \\ -x_2 + x_3 + 2x_4 &= -2. \end{aligned}$$

Hence, we obtain the equivalent system

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 5, \\ 2x_2 - 2x_3 - 4x_4 &= 1, \\ -x_2 + x_3 + 2x_4 &= -2. \end{aligned}$$

Next, multiply the third equation by -2:

$$2x_2 - 2x_3 - 4x_4 = 4.$$

Thus the system becomes

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 5, \\ 2x_2 - 2x_3 - 4x_4 &= 1, \\ 2x_2 - 2x_3 - 4x_4 &= 4. \end{aligned}$$

The last two equations imply $1 = 4$, which is a contradiction. Therefore, the system has *no solution.*

$$(d). \qquad \begin{cases} x_1 + 2x_2 + 2x_3 = 2 \\ x_1 + 8x_3 + 5x_4 = -6 \\ x_1 + x_2 + 5x_3 + 5x_4 = 3. \end{cases} \tag{4}$$

Xingjian's Solution:

We subtract the first equation from the second and from the third to eliminate x_1.

From $\text{Eq}_2 - \text{Eq}_1$:

$$\begin{aligned} (x_1 + 8x_3 + 5x_4) - (x_1 + 2x_2 + 2x_3) &= -6 - 2, \\ -2x_2 + 6x_3 + 5x_4 &= -8. \end{aligned} \tag{A}$$

From $\text{Eq}_3 - \text{Eq}_1$:

$$\begin{aligned} (x_1 + x_2 + 5x_3 + 5x_4) - (x_1 + 2x_2 + 2x_3) &= 3 - 2, \\ -x_2 + 3x_3 + 5x_4 &= 1. \end{aligned} \tag{B}$$

Now solve (A) and (B) for x_2 and x_4 in terms of x_3. From (B),

$$-x_2 + 3x_3 + 5x_4 = 1 \quad \Rightarrow \quad x_2 = 3x_3 + 5x_4 - 1.$$

Substitute this expression for x_2 into (A):

$$\begin{aligned} -2(3x_3 + 5x_4 - 1) + 6x_3 + 5x_4 &= -8, \\ -6x_3 - 10x_4 + 2 + 6x_3 + 5x_4 &= -8, \\ -5x_4 + 2 &= -8, \end{aligned}$$

so

$$-5x_4 = -10 \quad \Rightarrow \quad x_4 = 2.$$

Then from (B),

$$-x_2 + 3x_3 + 5 \cdot 2 = 1 \quad \Rightarrow \quad -x_2 + 3x_3 + 10 = 1 \quad \Rightarrow \quad x_2 = 3x_3 + 9.$$

Finally, substitute x_2 and x_4 into the first equation:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + 2(3x_3 + 9) + 2x_3 &= 2, \\ x_1 + 6x_3 + 18 + 2x_3 &= 2, \\ x_1 + 8x_3 + 18 &= 2, \end{aligned}$$

hence

$$x_1 = -8x_3 - 16.$$

Let t be a free parameter, with $t \in \mathbb{R}$ and $t = x_3$. Then the solution set is

$$(x_1, x_2, x_3, x_4) = (-8t - 16,\ 3t + 9,\ t,\ 2), \qquad t \in \mathbb{R}.$$

$$(e). \qquad \begin{cases} x_1 + 2x_2 - 4x_3 - x_4 + x_5 = 7 \\ -x_1 \qquad + 10x_3 - 3x_4 - 4x_5 = 16 \\ 2x_1 + 5x_2 - 5x_3 - 4x_4 - x_5 = 2 \\ 4x_1 + 11x_2 - 7x_3 - 10x_4 - 2x_5 = 7. \end{cases} \tag{5}$$

Xingjian's Solution: Add the first equation to the second to eliminate x_1:

$$\text{Eq}_1 + \text{Eq}_2: \quad (x_1 - x_1) + 2x_2 + (-4x_3 + 10x_3) + (-x_4 - 3x_4) + (x_5 - 4x_5) = 7 + 16,$$

so

$$2x_2 + 6x_3 - 4x_4 - 3x_5 = 23. \tag{A}$$

Next, subtract 2 times the first equation from the third:

$$2\text{Eq}_1: \quad 2x_1 + 4x_2 - 8x_3 - 2x_4 + 2x_5 = 14.$$

Then

$$\text{Eq}_3 - 2\text{Eq}_1: \quad (2x_1 - 2x_1) + (5x_2 - 4x_2) + (-5x_3 + 8x_3) + (-4x_4 + 2x_4) + (-x_5 - 2x_5) = 2 - 14,$$

giving

$$x_2 + 3x_3 - 2x_4 - 3x_5 = -12. \tag{B}$$

Now subtract 2 times (B) from (A):

$$(A) - 2(B): \quad \big(2x_2 + 6x_3 - 4x_4 - 3x_5\big) - \big(2x_2 + 6x_3 - 4x_4 - 6x_5\big) = 23 - 2(-12).$$

The left–hand side simplifies to

$$(2x_2 - 2x_2) + (6x_3 - 6x_3) + (-4x_4 + 4x_4) + (-3x_5 + 6x_5) = 3x_5,$$

and the right–hand side is

$$23 + 24 = 47,$$

so we obtain

$$3x_5 = 47.$$

This implies a specific value $x_5 = 47/3$. One can continue and check that, when we also incorporate the fourth equation, the augmented matrix reduces to a row of the form

$$0 = 1,$$

i.e. a contradiction.

Therefore, the system is *inconsistent* and has no solution.

$$(f). \qquad \begin{cases} x_1 + 2x_2 + 6x_3 = -1 \\ 2x_1 + x_2 + x_3 = 8 \\ 3x_1 + x_2 - x_3 = 15 \\ x_1 + 3x_2 + 10x_3 = -5. \end{cases} \tag{6}$$

Xingjian's Solution: Use the first equation to eliminate x_1 from the others. From the first equation,

$$x_1 = -1 - 2x_2 - 6x_3.$$

Substitute into the second:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 8, \\ 2(-1 - 2x_2 - 6x_3) + x_2 + x_3 &= 8, \\ -2 - 4x_2 - 12x_3 + x_2 + x_3 &= 8, \\ -2 - 3x_2 - 11x_3 &= 8, \end{aligned}$$

so

$$-3x_2 - 11x_3 = 10. \tag{1}$$

Now substitute into the third:

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 15, \\ 3(-1 - 2x_2 - 6x_3) + x_2 - x_3 &= 15, \\ -3 - 6x_2 - 18x_3 + x_2 - x_3 &= 15, \\ -3 - 5x_2 - 19x_3 &= 15, \end{aligned}$$

so

$$-5x_2 - 19x_3 = 18. \tag{2}$$

Next, substitute into the fourth:

$$\begin{aligned} x_1 + 3x_2 + 10x_3 &= -5, \\ (-1 - 2x_2 - 6x_3) + 3x_2 + 10x_3 &= -5, \\ -1 + x_2 + 4x_3 &= -5, \end{aligned}$$

so

$$x_2 + 4x_3 = -4 \quad \Rightarrow \quad x_2 = -4 - 4x_3. \tag{3}$$

Substitute (3) into (1):

$$-3(-4-4x_3)-11x_3=10 \Rightarrow 12+12x_3-11x_3=10 \Rightarrow 12+x_3=10 \Rightarrow x_3=-2.$$

Then from (3),

$$x_2=-4-4(-2)=-4+8=4.$$

Finally,

$$x_1=-1-2x_2-6x_3=-1-2\cdot 4-6(-2)=-1-8+12=3.$$

Thus the unique solution is

$$(x_1,x_2,x_3)=(3,4,-2).$$

Question 3: For each of the following lists of vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two.

(a). $(2,-1,0),(1,2,-3),(1,-3,2)$

Xingjian's Solution: we now determine whether

$$(2,-1,0)=a(1,2,-3)+b(1,-3,2)$$

for some $a,b\in\mathbb{R}$. This gives the system

$$\begin{aligned} a+b&=2,\\ 2a-3b&=-1,\\ -3a+2b&=0. \end{aligned}$$

Solving the first two equations yields $a=1$ and $b=1$. Substituting into the third equation gives

$$-3(1)+2(1)=-1\neq 0,$$

which is a contradiction.

Therefore, $(2,-1,0)$ *cannot* be expressed as a linear combination of $(1,2,-3)$ and $(1,-3,2)$.

(b). $(1,2,-3),(-3,2,1),(2,-1,-1)$

Xingjian's Solution: We ask whether

$$(1,2,-3)=a(-3,2,1)+b(2,-1,-1).$$

This gives the system

$$\begin{aligned} -3a+2b&=1,\\ 2a-b&=2,\\ a-b&=-3. \end{aligned}$$

From the second equation, $2a-b=2\Rightarrow b=2a-2$. Substitute into the third:

$$a-(2a-2)=-3\Rightarrow -a+2=-3\Rightarrow a=5.$$

Then $b = 2(5) - 2 = 8$.

Now check the first equation:

$$-3(5) + 2(8) = -15 + 16 = 1,$$

so it is satisfied.

Thus

$$(1, 2, -3) = 5(-3, 2, 1) + 8(2, -1, -1),$$

so the first vector *can* be expressed as a linear combination.

(c). $(3, 4, 1), (1, 2, -1), (-2, -1, 1)$

Xingjian's Solution: We ask whether

$$(3, 4, 1) = a(1, 2, -1) + b(-2, -1, 1).$$

So

$$\begin{aligned} a - 2b &= 3, \\ 2a - b &= 4, \\ -a + b &= 1. \end{aligned}$$

Solve the first two equations. From $a - 2b = 3$ we get $a = 3 + 2b$. Substitute into $2a - b = 4$:

$$2(3 + 2b) - b = 4 \Rightarrow 6 + 4b - b = 4 \Rightarrow 6 + 3b = 4 \Rightarrow 3b = -2 \Rightarrow b = -\frac{2}{3}.$$

Then $a = 3 + 2(-\frac{2}{3}) = 3 - \frac{4}{3} = \frac{5}{3}$.

Now check the third equation:

$$-a + b = -\frac{5}{3} - \frac{2}{3} = -\frac{7}{3} \neq 1.$$

Contradiction. Hence there is no solution for a, b, and $(3, 4, 1)$ *cannot* be expressed as a linear combination of the other two.

(d). $(-2, 0, 3), (1, 3, 0), (2, 4, -1)$

Xingjian's Solution: We ask whether

$$(-2, 0, 3) = a(1, 3, 0) + b(2, 4, -1).$$

This yields

$$\begin{aligned} a + 2b &= -2, \\ 3a + 4b &= 0, \\ -b &= 3. \end{aligned}$$

From the third equation, $b = -3$. Substitute into the first:

$$a + 2(-3) = -2 \Rightarrow a - 6 = -2 \Rightarrow a = 4.$$

Check the second equation:

$$3(4) + 4(-3) = 12 - 12 = 0,$$

which is satisfied. Thus

$$(-2, 0, 3) = 4(1, 3, 0) - 3(2, 4, -1),$$

so the first vector *is* a linear combination.

(e). $(5, 1, -5), (1, -2, -3), (-2, 3, -4)$

Xingjian's Solution: We ask whether

$$(5, 1, -5) = a(1, -2, -3) + b(-2, 3, -4).$$

So

$$\begin{aligned} a - 2b &= 5, \\ -2a + 3b &= 1, \\ -3a - 4b &= -5. \end{aligned}$$

From the first equation, $a = 5 + 2b$. Substitute into the second:

$$-2(5 + 2b) + 3b = 1 \Rightarrow -10 - 4b + 3b = 1 \Rightarrow -10 - b = 1 \Rightarrow b = -11.$$

Then $a = 5 + 2(-11) = 5 - 22 = -17$.

Check the third equation:

$$-3a - 4b = -3(-17) - 4(-11) = 51 + 44 = 95 \neq -5.$$

Contradiction. So there are no suitable a, b and $(5, 1, -5)$ *cannot* be expressed as a linear combination of the other two vectors.

(f). $(-2, 2, 2), (1, 2, -1), (-3, -3, 3)$

Xingjian's Solution: We ask whether

$$(-2, 2, 2) = a(1, 2, -1) + b(-3, -3, 3).$$

This gives

$$\begin{aligned} a - 3b &= -2, \\ 2a - 3b &= 2, \\ -a + 3b &= 2. \end{aligned}$$

Subtract the first equation from the second:

$$(2a - 3b) - (a - 3b) = 2 - (-2) \Rightarrow a = 4.$$

Then from $a - 3b = -2$ we get

$$4 - 3b = -2 \Rightarrow -3b = -6 \Rightarrow b = 2.$$

Check the third equation:

$$-a + 3b = -4 + 6 = 2,$$

which is satisfied. Thus

$$(-2, 2, 2) = 4(1, 2, -1) + 2(-3, -3, 3),$$

so the first vector *is* a linear combination.

Question 5: In each part, determine whether the given vector is in the span of S.

(a). $(2, -1, 1), \mathbf{S} = \{(1, 0, 2), (-1, 1, 1)\}$

Xingjian's Solution: We ask whether

$$(2, -1, 1) = a(1, 0, 2) + b(-1, 1, 1).$$

So

$$\begin{aligned} a - b &= 2, \\ 0 \cdot a + b &= -1, \\ 2a + b &= 1. \end{aligned}$$

From the second equation, $b = -1$. Then from the first,

$$a - (-1) = 2 \Rightarrow a + 1 = 2 \Rightarrow a = 1.$$

Check the third:

$$2(1) + (-1) = 1,$$

so it holds. Hence $(2, -1, 1)$ is in span(S).

(b). $(-1, 2, 1), \mathbf{S} = \{(1, 0, 2), (-1, 1, 1)\}$

Xingjian's Solution: We ask whether

$$(-1, 2, 1) = a(1, 0, 2) + b(-1, 1, 1).$$

This yields

$$\begin{aligned} a - b &= -1, \\ b &= 2, \\ 2a + b &= 1. \end{aligned}$$

From $b = 2$, the first equation gives $a - 2 = -1 \Rightarrow a = 1$. Then the third equation becomes

$$2(1) + 2 = 4 \neq 1.$$

Contradiction. So $(-1, 2, 1) \notin$ span(S).

(c). $(-1, -1, 1, 2), \mathbf{S} = \{(1, 0, 1, -1), (0, 1, 1, 1)\}$

Xingjian's Solution: We ask whether

$$(-1, -1, 1, 2) = a(1, 0, 1, -1) + b(0, 1, 1, 1).$$

This gives

$$\begin{aligned} a &= -1, \\ b &= -1, \\ a + b &= 1, \\ -a + b &= 2. \end{aligned}$$

From the first two equations, $a = -1$ and $b = -1$, so $a + b = -2$, which does *not* equal 1. Thus the system is inconsistent and the vector is not in the span:

$$(-1, -1, 1, 2) \notin \text{span}(S).$$

(d). $(2, -1, -1, 3), \mathbf{S} = \{(1, 0, 1, -1), (0, 1, 1, 1)\}$
Xingjian's Solution: We ask whether

$$(2, -1, -1, 3) = a(1, 0, 1, -1) + b(0, 1, 1, 1).$$

So

$$\begin{aligned} a &= 2, \\ b &= -1, \\ a + b &= -1, \\ -a + b &= 3. \end{aligned}$$

From $a = 2$ and $b = -1$, we get $a + b = 1$, which does not equal -1. Again we get a contradiction. Thus

$$(2, -1, -1, 3) \notin \text{span}(S).$$

(e). $-x^3 + 2x^2 + 3x + 3, \qquad S = \{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\}$
Xingjian's Solution: We check whether

$$-x^3 + 2x^2 + 3x + 3$$

lies in the span of

$$S = \{p_1, p_2, p_3\} = \{x^3 + x^2 + x + 1,\ x^2 + x + 1,\ x + 1\}.$$

We want

$$-x^3 + 2x^2 + 3x + 3 = ap_1 + bp_2 + cp_3.$$

Write everything in the standard basis $1, x, x^2, x^3$. Then

$$a(x^3 + x^2 + x + 1) + b(x^2 + x + 1) + c(x + 1)$$

has coefficients:

$$\begin{aligned} x^3 &: a, \\ x^2 &: a + b, \\ x^1 &: a + b + c, \\ 1 &: a + b + c. \end{aligned}$$

We match these with $-1, 2, 3, 3$ respectively:

$$\begin{aligned} a &= -1, \\ a + b &= 2, \\ a + b + c &= 3, \\ a + b + c &= 3. \end{aligned}$$

From $a = -1$, we get $-1+b = 2 \Rightarrow b = 3$. Then $a+b+c = -1+3+c = 2+c = 3 \Rightarrow c = 1$. All conditions agree.

So

$$-x^3 + 2x^2 + 3x + 3 = -p_1 + 3p_2 + p_3,$$

and the polynomial is in span(S).

(f). $2x^3 - x^2 + x + 3, \qquad S = \{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\}$

Xingjian's Solution: Now check

$$2x^3 - x^2 + x + 3 = ap_1 + bp_2 + cp_3.$$

As above, the right–hand side has coefficients

$$(x^3, x^2, x, 1) : \ (a, \ a + b, \ a + b + c, \ a + b + c).$$

We equate to $(2, -1, 1, 3)$:

$$\begin{aligned} a &= 2, \\ a + b &= -1, \\ a + b + c &= 1, \\ a + b + c &= 3. \end{aligned}$$

From $a = 2$ and $a + b = -1$ we get $b = -3$. Then $a + b + c = 1$ implies $2 - 3 + c = 1 \Rightarrow -1 + c = 1 \Rightarrow c = 2$. But the constant term requires $a + b + c = 3$, which would give $2 - 3 + 2 = 1$, not 3. Contradiction.

Therefore, $2x^3 - x^2 + x + 3$ is *not* in span(S).

(g). $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}, \qquad S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

Xingjian's Solution: We ask whether

$$M = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$$

is in the span of

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

We seek a, b, c with

$$aA + bB + cC = M.$$

Compute the left–hand side:

$$aA + bB + cC = \begin{pmatrix} a & 0 \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a + c & b + c \\ -a & b \end{pmatrix}.$$

Set this equal to M:

$$\begin{pmatrix} a + c & b + c \\ -a & b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}.$$

Thus

$$\begin{aligned} a + c &= 1, \\ b + c &= 2, \\ -a &= -3, \\ b &= 4. \end{aligned}$$

From $-a = -3$ we get $a = 3$, and from $b = 4$ we have $b = 4$. Then

$$a + c = 1 \Rightarrow 3 + c = 1 \Rightarrow c = -2,$$

and $b + c = 4 + (-2) = 2$, which agrees.

So

$$M = 3A + 4B - 2C,$$

and $M \in \text{span}(S)$.

(h). $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

Xingjian's Solution: Now check whether

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is in the span of the same A, B, C.

Again, $aA + bB + cC$ has the form

$$\begin{pmatrix} a + c & b + c \\ -a & b \end{pmatrix}.$$

We want

$$\begin{pmatrix} a + c & b + c \\ -a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned} a + c &= 1, \\ b + c &= 0, \\ -a &= 0, \\ b &= 1. \end{aligned}$$

From $-a = 0$ we get $a = 0$, and from $b = 1$ we have $b = 1$. Then $a + c = 1$ gives $c = 1$, but $b + c = 1 + 1 = 2$, which does *not* equal 0. Contradiction.

Therefore $I \notin \text{span}(S)$.

Question 6: Show that the vectors $(1, 1, 0), (1, 0, 1)$, and $(0, 1, 1)$ generate $\mathbf{F}^3$.

Xingjian's Solution: We must show that any vector $(x, y, z) \in \mathbf{F}^3$ can be written as a linear combination of

$$\mathbf{v}_1 = (1, 1, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (0, 1, 1).$$

Suppose

$$(x, y, z) = a\,\mathbf{v}_1 + b\,\mathbf{v}_2 + c\,\mathbf{v}_3 = a(1, 1, 0) + b(1, 0, 1) + c(0, 1, 1).$$

Then

$$a(1, 1, 0) + b(1, 0, 1) + c(0, 1, 1) = (a + b,\ a + c,\ b + c).$$

So we must solve

$$(a + b,\ a + c,\ b + c) = (x, y, z),$$

i.e.

$$\begin{aligned} a + b &= x, \\ a + c &= y, \\ b + c &= z. \end{aligned}$$

Adding the first two equations and subtracting the third:

$$(a + b) + (a + c) - (b + c) = x + y - z \Rightarrow 2a = x + y - z.$$

Assuming 2 is invertible in $\mathbf{F}$, we get

$$a = \frac{x + y - z}{2}.$$

Then from $a + b = x$ we have $b = x - a$, and from $a + c = y$ we get $c = y - a$. Thus for every (x, y, z), there exist a, b, c with

$$(x, y, z) = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3.$$

Hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbf{F}^3$, i.e. these vectors generate $\mathbf{F}^3$.

Question 12: Show that a subset $\mathbf{W}$ of a vector space $\mathbf{V}$ is a subspace of $\mathbf{V}$ if and only if $\text{span}(\mathbf{W}) = \mathbf{W}$

Xingjian's Solution: We must show: for $W \subset V$, W is a subspace of V if and only if $\text{span}(W) = W$.

($\Rightarrow$) Assume W is a subspace of V. By definition, $\text{span}(W)$ is the set of all finite linear combinations of elements of W. Since W is a subspace, it is closed under addition and scalar multiplication, so every linear combination of elements of W is still in W. Thus $\text{span}(W) \subseteq W$.

On the other hand, every $w \in W$ can be written as $1 \cdot w$, which is a linear combination of elements of W, so $w \in \text{span}(W)$, and $W \subseteq \text{span}(W)$. Therefore $\text{span}(W) = W$.

($\Leftarrow$) Now assume $\text{span}(W) = W$. We know that $\text{span}(W)$ is always a subspace of V (this is a standard lemma: the span of any subset is a subspace). Since $W = \text{span}(W)$, it follows that W is also a subspace of V.

Thus, W is a subspace of V if and only if $\text{span}(W) = W$.

Xingjian Xu
xingjianxu@ufl.edu
LIT 453
Webpage: https://people.clas.ufl.edu/xingjianxu/mas4105-26springs/
Office Hours: Tuesday, Periods 3 and 5 on LIT453;