**Question 1:** Label the following statement as true or false.

(a). If $\mathbf{V}$ is a vector space and $\mathbf{W}$ is a subset of $\mathbf{V}$ that is a vector space, then $\mathbf{W}$ is a subspace of $\mathbf{V}$.

(b). The empty set is a subspace of every vector space.

(c). If $\mathbf{V}$ is a vector space other than the zero vector space, then $\mathbf{V}$ contains a subspace W such that $\mathbf{W} \neq \mathbf{V}$.

(d). The intersection of any two subsets of $\mathbf{V}$ is a subspace of $\mathbf{V}$.

(e). An $n \times n$ diagonal matrix can never have more than $n$ nonzero entries.

(f). The trace of a square matrix is the product of its diagonal entries.

(g). Let $\mathbf{W}$ be the $xy$-plane in $\mathbb{R}^3$; that is,

$$W = (a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}.$$

Then $W = \mathbb{R}^2$.

*Xingjian's Solution:* (a). **False** This should make sure that the field and the operations of $\mathbf{V}$ and $\mathbf{W}$ are the same. Otherwise, for example, $\mathbf{V} = \mathbb{R}$ and $\mathbf{W} = \mathbb{Q}$ respectably. Then $\mathbf{W}$ is a vector space over $\mathbb{Q}$ but not a space over $\mathbb{R}$ and so not a subspace of $\mathbf{V}$ .

(b). **False**. $\emptyset$ does not contain any vector. It breaks the zero vector in subspace.

(c). **True**.

(d). **False.** Take $V = \mathbb{R}^2$, $A = \{(1, 0)\}$ and $B = \{(0, 1)\}$. Then $A \cap B = \emptyset$, which is not a subspace.

(e). **True**.

(f). **False.** The trace of a square matrix is the sum of its diagonal entries, not the product.

(g). **False.** every element of $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$ has *three* components and must have third coordinate equal to 0. There is no element of $W$ that is equal to $(1, 2)$, since $(1, 2)$ is not a triple. but You indeed can say they are *isomorphic*. **Xingjian comment:** Isomorphism always show up in abstract algebra or 5000 level and above algebra course. It is fine you didn't hear about this.

**Question 5:** Prove that $A + A^t$ is symmetric for any square matrix $A$.

*Xingjian's Solution:*

*Proof.* Recall that for any square matrix $A$ is symmetric if $A^t = A$. We have

$$(A + A^t)^t = A^t + (A^t)^t = A^t + A,$$

since the transpose is linear and $(A^t)^t = A$. Thus $(A + A^t)^t = A + A^t$, so $A + A^t$ is symmetric. □

**Question 8:** Determine whether the following sets are subspaces of $\mathbb{R}^3$ under operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers

(a). $\mathbf{W}_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

*Xingjian's Solution:* Every vector in $W_1$ has the form $(3t, t, -t)$ for some $t \in \mathbb{R}$.

**(Zero vector)**: for $t = 0$ we get $(0, 0, 0) \in W_1$.

**(Closed under Addition)**: if $(3t_1, t_1, -t_1)$ and $(3t_2, t_2, -t_2)$ are in $W_1$, then their sum is

$$(3t_1, t_1, -t_1) + (3t_2, t_2, -t_2) = (3(t_1 + t_2), t_1 + t_2, -(t_1 + t_2)),$$

which is still of the required form.

**(Closed under Scalar multiplication)**: for $\lambda \in \mathbb{R}$, $\lambda(3t, t, -t) = (3(\lambda t), \lambda t, -\lambda t) \in W_1$.

Hence $W_1$ is a subspace of $\mathbb{R}^3$. (b). $\mathbf{W}_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$

*Xingjian's Solution:* Check the zero vector: Suppose that $(0, 0, 0) \in W_2$ , it requires $0 = 0 + 2$, which is impossible. This is a contradiction. Thus, $(0, 0, 0) \notin W_2$, so $W_2$ is not a subspace.

(c). $\mathbf{W}_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$

*Xingjian's Solution:*

**(Zero vector)**: $2 \cdot 0 - 7 \cdot 0 + 0 = 0$, so $0 \in W_3$.

**(Closed under Addition)**: If $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in W_3$, then $2u_1 - 7u_2 + u_3 = 0$ and $2v_1 - 7v_2 + v_3 = 0$. For $\mathbf{u} + \mathbf{v}$ we have

$$2(u_1 + v_1) - 7(u_2 + v_2) + (u_3 + v_3) = (2u_1 - 7u_2 + u_3) + (2v_1 - 7v_2 + v_3) = 0 + 0 = 0.$$

So $\mathbf{u} + \mathbf{v} \in W_3$.

**(Closed under Scalar multiplication)**: For $\lambda \in \mathbb{R}$,

$$2(\lambda a_1) - 7(\lambda a_2) + (\lambda a_3) = \lambda(2a_1 - 7a_2 + a_3) = \lambda \cdot 0 = 0,$$

so $\lambda \mathbf{a} \in W_3$ whenever $\mathbf{a} \in W_3$.

Hence $W_3$ is a subspace.

(d). $\mathbf{W}_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$

*Xingjian's Solution:* Exactly the same reasoning as in part (c) applies: this is again the solution set of a homogeneous linear equation, so it contains 0 and is closed under addition and scalar multiplication. Therefore $W_4$ is a subspace. (You can try it for practice.)

(e). $\mathbf{W}_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$

*Xingjian's Solution:*

Check the zero vector: for $(0, 0, 0)$ to lie in $W_5$ we would need $0 + 0 - 0 = 1$, which is false. Thus $(0, 0, 0) \notin W_5$, so $W_5$ is not a subspace.

(f). $\mathbf{W}_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 - 6a_3^2 = 0\}$

*Xingjian's Solution:*

Counterexample for (closed under addition): We have both $(\sqrt{3}, \sqrt{5}, 0)$ and $(0, \sqrt{6}, \sqrt{3})$ two elements of $W_6$ but their sum $(\sqrt{3}, \sqrt{5} + \sqrt{6}, \sqrt{3})$ is not an element of $W_6$.

**Question 11:** Is the set $\mathbf{W} = \{f(x) \in \mathbf{P}(F) : f(x) = 0$ or $f(x)$has degree $n\}$ a subspace of $\mathbf{P}(F)$ if $n \geq 1$? Justify your answer.

*Xingjian's Solution:*

First note that $0 \in W$ by definition, so the set is nonempty and contains the zero vector.

However, $W$ is not closed under addition. Take

$$p(x) = x^n, \qquad q(x) = -x^n.$$

Both $p$ and $q$ are polynomials of degree $n$, so $p, q \in W$. But

$$p(x) + q(x) = x^n + (-x^n) = 0,$$

which is a nonzero polynomial of degree $-1 \neq n$. Thus $p + q \notin W$, so $W$ is not closed under addition and therefore is not a subspace of $\mathbf{P}(F)$.

**Question 12:** An $m \times n$ matrix $A$ is called upper triangular if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $M_{mn}(F)$.

*Xingjian's Solution:*

*Proof.* Let $U$ be the set of all $n \times n$ upper triangular matrices over $F$.

**(Zero matrix)**: the zero matrix has all entries equal to 0, so in particular $A_{ij} = 0$ for all $i > j$. Hence $0 \in U$.

**(Closed under Addition)**: let $A, B \in U$. Then for all $i > j$ we have $A_{ij} = 0$ and $B_{ij} = 0$. Therefore

$$(A + B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0$$

for all $i > j$, so $A + B$ is also upper triangular, that is, $A + B \in U$.

**(Closed under Scalar multiplication)**: let $\lambda \in F$ and $A \in U$. For $i > j$, $(\lambda A)_{ij} = \lambda A_{ij} = \lambda \cdot 0 = 0$, so $\lambda A$ is upper triangular and lies in $U$.

Thus $U$ contains 0 and is closed under addition and scalar multiplication. Therefore $U$ is a subspace of $M_n(F)$. □

**Question 13:** Let $S$ be a nonempty set and $F$ a field. Prove that for any $s_0 \in S$, $\{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$, is a subspace of $\mathcal{F}(S, F)$.

*Xingjian's Solution:*

*Proof.* **(Zero function)**: the zero function $0 : S \to F$ is defined by $0(s) = 0$ for all $s \in S$. In particular $0(s_0) = 0$, so $0 \in W$.

**(Closed under Addition)**: Let $f, g \in W$. Then $f(s_0) = 0$ and $g(s_0) = 0$. For their sum we have $(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0$, so $f + g \in W$.

**(Closed under Scalar multiplication)**: Let $\lambda \in F$ and $f \in W$. Then $(\lambda f)(s_0) = \lambda f(s_0) = \lambda \cdot 0 = 0$, so $\lambda f \in W$.

Therefore $W$ is a subspace of $\mathcal{F}(S, F)$. □

**Question 15:** Is the set of all differentiable real-valued functions defined on $\mathbb{R}$ a subspace of $\mathbf{C}(\mathbb{R})$? Justify your answer.

*Xingjian's Solution:* Yes, I will prove in the following:

*Proof.* First, we need to know every differentiable function on $\mathbb{R}$ is continuous on $\mathbb{R}$, so $D \subseteq C(\mathbb{R})$.

**(Zero function)**: the zero function $f(x) \equiv 0$ is differentiable (its derivative is identically 0), so $0 \in D$.

**(Closed under Addition)**: If $f, g \in D$, then both are differentiable on $\mathbb{R}$, and the sum $f + g$ is differentiable on $\mathbb{R}$ (with derivative $f' + g'$). Thus $f + g \in D$.

**(Closed under Scalar multiplication)**: If $\lambda \in \mathbb{R}$ and $f \in D$, then $\lambda f$ is differentiable on $\mathbb{R}$ with derivative $\lambda f'$, so $\lambda f \in D$.

Therefore $D$ contains the zero function and is closed under addition and scalar multiplication, so $D$ is a subspace of $C(\mathbb{R})$. □

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