Question 1: Label the following statements as true or false:

(a). Every vector space contains a zero vector.

(b). A vector space may have more than one zero vector.

(c). In any vector space, $ax = bx$ implies that $a = b$.

(d). In any vector space, $ax = ay$ implies that $x = y$.

(e). A vector in $\mathbf{F}^n$ may be regarded as a matrix in $\mathbf{M}_{n\times 1}(F)$.

(f). An $m \times n$ matrix has m columns and n rows.

(g). In $\mathbf{P}(F)$, only polynomials of the same degree may be added.

(h). If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.

(i). If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.

(j). A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero.

(k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if they have the same value at each element of S.

Xingjian's solution (a). True. It is the condition VS(3).

(b). False, only unique zero vector. Suppose x, y are both zero vectors. Then by definition $x = x+ = y = y$ by VS(3). It is contradiction.

(c). False. Let e be the zero vector, we have $1e = 2e$.

(d). False. It will be false when $a = 0$.

(e). True.

(f). False. It has m rows and n columns.

(g). False. Polynomial with degree less than n can be added.

(h). False. Counterexample for $x + (-1)x = 0$.

(i). True.

(j). True.

(k). True.

Question 7: Let $S = \{0, 1\}$ and $F = \mathbb{R}$. In $\mathcal{F}(S, \mathbb{R})$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1, g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.

Xingjian's solution: (a). We show that $f = g$ in $\mathcal{F}(S, \mathbb{R})$.

Since $S = \{0, 1\}$, it suffices to verify that

$$f(t) = g(t) \quad \text{for all } t \in S.$$

For $t = 0$,

$$f(0) = 2(0) + 1 = 1, \qquad g(0) = 1 + 4(0) - 2(0)^2 = 1.$$

For $t = 1$,

$$f(1) = 2(1) + 1 = 3, \qquad g(1) = 1 + 4(1) - 2(1)^2 = 3.$$

Thus $f(t) = g(t)$ for all $t \in S$, and hence

$$f = g \quad \text{in } \mathcal{F}(S, \mathbb{R}).$$

(b). We show that $f + g = h$ in $\mathcal{F}(S, \mathbb{R})$.

For $t = 0$,

$$(f+g)(0) = f(0) + g(0) = 1 + 1 = 2, \qquad h(0) = 5^0 + 1 = 2.$$

For $t = 1$,

$$(f+g)(1) = f(1) + g(1) = 3 + 3 = 6, \qquad h(1) = 5^1 + 1 = 6.$$

Hence $(f+g)(t) = h(t)$ for all $t \in S$, and therefore

$$f + g = h \quad \text{in } \mathcal{F}(S, \mathbb{R}).$$

Question 11: Let $\mathbf{V} = \{0\}$ consist of a single vector 0 and define $0+0 = 0$ and $c0 = 0$ for each scalar c in F. Prove that $\mathbf{V}$ is a vector space over F. ($\mathbf{V}$ is called the zero vector space.)

Xingjian's solution. We verify the vector space axioms.

(1) *Closure under addition.* Since $\mathbf{V}$ contains only the vector $\mathbf{0}$,

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \in \mathbf{V}.$$

(2) *Closure under scalar multiplication.* For any $c \in F$,

$$c\,\mathbf{0} = \mathbf{0} \in \mathbf{V}.$$

(3) *Additive identity.* The vector $\mathbf{0}$ satisfies

$$\mathbf{0} + \mathbf{0} = \mathbf{0},$$

so $\mathbf{0}$ is the additive identity in $\mathbf{V}$.

(4) *Additive inverse.* The additive inverse of $\mathbf{0}$ is $\mathbf{0}$, since

$$\mathbf{0} + \mathbf{0} = \mathbf{0}.$$

(5) *All remaining axioms.* Associativity, commutativity of addition, and the distributive properties all hold trivially, since every expression involves only the single vector $\mathbf{0}$.

Therefore, all vector space axioms are satisfied, and $\mathbf{V}$ is a vector space over F. $\square$

Question 12: A real-valued function f defined on the real line is called an even function if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Xingjian's solution Let

$$E := \{f : \mathbb{R} \to \mathbb{R} \mid f(-t) = f(t) \text{ for all } t \in \mathbb{R}\}$$

be the set of all even real-valued functions on $\mathbb{R}$. We verify that $(E, +, \cdot)$ satisfies the vector space axioms VS1–VS8 over $\mathbb{R}$.

VS1 (closure under addition). Let $f, g \in E$. For any $t \in \mathbb{R}$,

$$(f+g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f+g)(t),$$

so $f + g$ is even and hence $f + g \in E$.

VS2 (commutativity of addition). For $f, g \in E$ and all $t \in \mathbb{R}$,

$$(f+g)(t) = f(t) + g(t) = g(t) + f(t) = (g+f)(t),$$

so $f + g = g + f$.

VS3 (associativity of addition). For $f, g, h \in E$ and all $t \in \mathbb{R}$,

$$\big((f+g)+h\big)(t) = (f+g)(t)+h(t) = (f(t)+g(t))+h(t) = f(t)+(g(t)+h(t)) = \big(f+(g+h)\big)(t),$$

so $(f + g) + h = f + (g + h)$.

VS4 (additive identity). Define $0 : \mathbb{R} \to \mathbb{R}$ by $0(t) = 0$ for all t. Then $0(-t) = 0 = 0(t)$, so $0 \in E$. For any $f \in E$ and all $t \in \mathbb{R}$,

$$(f+0)(t) = f(t) + 0 = f(t),$$

so $f + 0 = f$.

VS5 (additive inverses). Let $f \in E$ and define $-f$ by $(-f)(t) := -f(t)$. Then for all $t \in \mathbb{R}$,

$$(-f)(-t) = -f(-t) = -f(t) = (-f)(t),$$

so $-f \in E$. Moreover, for all t,

$$(f + (-f))(t) = f(t) + (-f(t)) = 0 = 0(t),$$

so $f + (-f) = 0$.

VS6 (closure under scalar multiplication). Let $c \in \mathbb{R}$ and $f \in E$. For all $t \in \mathbb{R}$,

$$(cf)(-t) = cf(-t) = cf(t) = (cf)(t),$$

so cf is even and hence $cf \in E$.

VS7 (distributivity of scalar over vector addition). Let $c \in \mathbb{R}$ and $f, g \in E$. For all $t \in \mathbb{R}$,

$$c(f+g)(t) = c\big((f+g)(t)\big) = c\big(f(t)+g(t)\big) = cf(t) + cg(t) = (cf+cg)(t),$$

so $c(f + g) = cf + cg$.

VS8 (distributivity and compatibility of scalars). Let $c, d \in \mathbb{R}$ and $f \in E$. Then for all $t \in \mathbb{R}$,

$$(c+d)f(t) = (c+d)f(t) = cf(t) + df(t) = (cf + df)(t),$$

so $(c+d)f = cf + df$. Also,

$$c(df)(t) = c\big(df(t)\big) = (cd)f(t),$$

so $c(df) = (cd)f$, and

$$1 \cdot f(t) = f(t) \quad \forall t,$$

so $1 \cdot f = f$.

All eight axioms VS1–VS8 hold for E with the given operations. Therefore, the set of even functions on $\mathbb{R}$ is a vector space over $\mathbb{R}$.

Question 13: Let $\mathbf{V}$ denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of $\mathbf{V}$ and $c \in \mathbb{R}$, define:

$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c(a_1, a_2) = (ca_1, a_2)$.

Is $\mathbf{V}$ a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Xingjian's solution. We show that $\mathbf{V}$ is *not* a vector space by checking the additive inverse axiom.

First observe that there is an additive identity in $\mathbf{V}$. Suppose (e_1, e_2) is an additive identity. Then for all $(a_1, a_2) \in \mathbf{V}$,

$$(a_1, a_2) + (e_1, e_2) = (a_1, a_2) \quad \text{and} \quad (e_1, e_2) + (a_1, a_2) = (a_1, a_2).$$

From

$$(a_1, a_2) + (e_1, e_2) = (a_1 + e_1,\ a_2 e_2) = (a_1, a_2)$$

we obtain

$$a_1 + e_1 = a_1 \Rightarrow e_1 = 0, \qquad a_2 e_2 = a_2 \text{ for all } a_2 \in \mathbb{R},$$

which forces $e_2 = 1$. The same conditions arise from $(e_1, e_2) + (a_1, a_2)$, so the unique additive identity is $(0, 1)$.

Now consider the vector $(0, 0) \in \mathbf{V}$. If $\mathbf{V}$ were a vector space, then $(0, 0)$ would have an additive inverse (x_1, x_2) such that

$$(0, 0) + (x_1, x_2) = (0, 1).$$

By the definition of addition,

$$(0, 0) + (x_1, x_2) = (0 + x_1,\ 0 \cdot x_2) = (x_1, 0).$$

Thus we would need

$$(x_1, 0) = (0, 1),$$

which is impossible. Hence $(0, 0)$ has no additive inverse.

Therefore the additive inverse axiom fails, and $\mathbf{V}$ is not a vector space over $\mathbb{R}$ with these operations.

For scalar multiplication failure, we know scalar multiplication must satisfy

$$1v = v \quad \text{for all } v \in \mathbf{V}.$$

Let $(a_1, a_2) \in \mathbf{V}$. Then

$$1(a_1, a_2) = (1 \cdot a_1,\ a_2) = (a_1,\ a_2),$$

so the scalar identity axiom holds.

However, scalar multiplication must also satisfy

$$(c + d)v = cv + dv \quad \text{for all } c, d \in \mathbb{R}.$$

Let $c = d = 1$ and $v = (0, 2)$. Then

$$(c + d)v = 2(0, 2) = (0, 2),$$

while

$$cv + dv = (1(0, 2)) + (1(0, 2)) = (0, 2) + (0, 2) = (0, 4).$$

Since

$$(0, 2) \neq (0, 4),$$

the distributive axiom $(c + d)v = cv + dv$ fails.

Therefore, $\mathbf{V}$ is not a vector space over $\mathbb{R}$

□

Question 18: Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in \mathbf{V}$ and $c \in \mathbb{R}$, define $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$.

Is $\mathbf{V}$ a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Xingjian's solution. We show that $\mathbf{V}$ is *not* a vector space by showing that no additive identity exists.

Suppose (e_1, e_2) is an additive identity in $\mathbf{V}$. Then for all $(a_1, a_2) \in \mathbf{V}$ we must have

$$(a_1, a_2) + (e_1, e_2) = (a_1, a_2) \quad \text{and} \quad (e_1, e_2) + (a_1, a_2) = (a_1, a_2).$$

From

$$(a_1, a_2) + (e_1, e_2) = (a_1 + 2e_1,\ a_2 + 3e_2) = (a_1, a_2),$$

we obtain

$$a_1 + 2e_1 = a_1 \Rightarrow 2e_1 = 0 \Rightarrow e_1 = 0,$$
$$a_2 + 3e_2 = a_2 \Rightarrow 3e_2 = 0 \Rightarrow e_2 = 0.$$

So any additive identity must be $(0, 0)$.

Now check whether $(0, 0)$ actually behaves as an identity on the left:

$$(0, 0) + (a_1, a_2) = (0 + 2a_1,\ 0 + 3a_2) = (2a_1, 3a_2).$$

For $(0, 0)$ to be an additive identity, we would need

$$(2a_1, 3a_2) = (a_1, a_2) \quad \text{for all } (a_1, a_2) \in \mathbf{V},$$

which implies

$$2a_1 = a_1, \quad 3a_2 = a_2 \quad \text{for all } a_1, a_2 \in \mathbb{R}.$$

This would force $a_1 = 0$ and $a_2 = 0$ for all vectors, which is false.

Thus there is no vector in $\mathbf{V}$ that serves as an additive identity for all elements of $\mathbf{V}$. Since the additive identity axiom fails, $\mathbf{V}$ is not a vector space over $\mathbb{R}$ with these operations.

In a vector space, scalar multiplication must satisfy

$$c(u+v) = cu + cv \quad \text{for all } c \in \mathbb{R},\ u, v \in \mathbf{V}.$$

Let $u = (1,1)$, $v = (1,1)$, and $c = 1$. Then

$$u + v = (1 + 2 \cdot 1,\ 1 + 3 \cdot 1) = (3,4).$$

Hence,

$$c(u+v) = 1(3,4) = (3,4).$$

On the other hand,

$$cu = (1,1), \qquad cv = (1,1),$$

and thus

$$cu + cv = (1,1) + (1,1) = (1 + 2 \cdot 1,\ 1 + 3 \cdot 1) = (3,4).$$

So distributivity holds in this case. However, scalar multiplication must also satisfy

$$(c+d)v = cv + dv \quad \text{for all } c, d \in \mathbb{R}.$$

Let $v = (1,1)$ and choose $c = d = 1$. Then

$$(c+d)v = 2(1,1) = (2,2),$$

while

$$cv + dv = (1,1) + (1,1) = (3,4).$$

Since

$$(2,2) \neq (3,4),$$

the axiom $(c+d)v = cv + dv$ fails.

Therefore, $\mathbf{V}$ is not a vector space over $\mathbb{R}$. □

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