For a fixed $a \in \mathbb{R}$, determine the dimension of the subspace of $\mathbf{P}_n(\mathbb{R})$ defined by $\{f \in \mathbf{P}_n(\mathbb{R}) : f(a) = 0\}$.

Xingjian's Solution: We consider the subspace

$$U = \{f \in \mathbf{P}_n(\mathbb{R}) : f(a) = 0\},$$

where $a \in \mathbb{R}$ is fixed.

Proof. Recall that $\mathbf{P}_n(\mathbb{R})$ consists of all real polynomials of degree at most n and has dimension $n+1$, with standard basis

$$\{1, x, x^2, \ldots, x^n\}.$$

First, note that if $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$. Thus every polynomial $f \in U$ can be written in the form

$$f(x) = (x-a)g(x),$$

where $g(x)$ is a polynomial of degree at most $n-1$.

Conversely, for any polynomial $g \in \mathbf{P}_{n-1}(\mathbb{R})$ of degree at most $n-1$, the polynomial $f(x) = (x-a)g(x)$ satisfies $f(a) = 0$, so $f \in U$. Hence every element of U arises in this way.

Now consider the set

$$\mathcal{B} = \{(x-a), (x-a)x, (x-a)x^2, \ldots, (x-a)x^{n-1}\}.$$

Each polynomial in $\mathcal{B}$ belongs to U, and there are exactly n of them.

(I).$a = 0$,

$$\mathcal{B} = \{x, x^2, x^3, \ldots, x^n\}.$$

Then it is a basis; so $\dim U = n$ as a trivial case.

(II). if $a \neq 0$, We claim that $\mathcal{B}$ is a basis for U.

Spanning. Let $f \in U$. Then $f(x) = (x-a)g(x)$ for some $g(x) \in \mathbf{P}_{n-1}(\mathbb{R})$. Since $\{1, x, x^2, \ldots, x^{n-1}\}$ is a basis for $\mathbf{P}_{n-1}(\mathbb{R})$, we may write

$$g(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}.$$

Thus

$$f(x) = (x-a)(c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}),$$

which is a linear combination of the elements of $\mathcal{B}$. Hence $\mathcal{B}$ spans U.

Linear independence. Suppose

$$c_0(x-a) + c_1(x-a)x + \cdots + c_{n-1}(x-a)x^{n-1} = 0.$$

Factoring out $(x-a)$ gives

$$(x-a)\big(c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}\big) = 0.$$

Since $(x-a)$ is not the zero polynomial, the factor in parentheses must be the zero polynomial. Thus

$$c_0 = c_1 = \cdots = c_{n-1} = 0.$$

Hence the set $\mathcal{B}$ is linearly independent.

Therefore $\mathcal{B}$ is a basis for U, and since it has n elements, we conclude that

$$\dim U = n.$$

□

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