UF MAS 4105 **Exam 3**

Name: _____

- Explain your work (efficiently); partial credit is available.
- No notes, books, calculators, or other electronic devices are permitted.
- Problem 6 is the bonus question which might help your grades in Exam 3.
- Please sign below to indicate you accept the following statement:

  "I will not give, receive, or use any unauthorized assistance."

Signature: _____

Problem	**Total Points**	**Score**
1	10	
2	10	
3	10	
4	10	
5	10	
6	**Bonus**	
Total	**50**	

1 For (1)–(5), determine whether each statement is **True** or **False**. For (6)–(10), give a brief definition of the terminology.

Throughout this problem, V and W are vector spaces with ordered finite bases α and β, respectively, and $T : V \to W$ is a linear transformation. Matrices A and B are square matrices of appropriate size.

(1). If $m = \dim(V)$ and $n = \dim(W)$, then the matrix representation $[T]_\alpha^\beta$ is an $n \times m$ matrix.

(2). If T is linear, then T preserves vector addition and scalar multiplication.

(3). If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.

(4). If $A^2 = O$, then $A = O$, where O denotes the zero matrix.

(5). $\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\alpha^\beta$.

(6). Give a brief definition of: V is isomorphic to W.

(7). State the Dimension Theorem.

(8). What does it mean for a linear transformation T to be one-to-one?

(9). What does it mean for a linear transformation T to be onto?

(10). Give the definition of the inverse of a linear transformation T.

Solutions:

1. **True.** If $\dim(V) = m$ and $\dim(W) = n$, then $T : V \to W$ takes vectors in V to vectors in W. Therefore $[T]_\alpha^\beta$ has one column for each basis vector of V and one row for each basis vector of W, so it is an $n \times m$ matrix.

2. **True.** By definition of linearity,

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$$

for all $u, v \in V$ and all scalars c. So T preserves vector addition and scalar multiplication.

3. **False.** A linear transformation does not always send linearly independent sets to linearly independent sets. This only happens when T is one-to-one.

For example, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the zero transformation:

$$T(x, y) = (0, 0).$$

Then $\{(1, 0)\}$ is linearly independent in $\mathbb{R}^2$, but its image is

$$\{T(1, 0)\} = \{(0, 0)\},$$

and $\{(0, 0)\}$ is not linearly independent.

4. **False.** It is possible for $A^2 = O$ without $A = O$.

   For example,
   $$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq O,$$
   but
   $$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

5. **False.** The expression $[T^{-1}]_{\alpha}^{\beta}$ does not make sense in general, because if $T : V \to W$, then $T^{-1} : W \to V$. So the correct matrix for T^{-1} should reverse the bases: $[T^{-1}]_{\beta}^{\alpha}$. When T is invertible, the correct formula is $\left([T]_{\alpha}^{\beta}\right)^{-1} = [T^{-1}]_{\beta}^{\alpha}$.

6. **Definition:** V is **isomorphic** to W if there exists a linear transformation
   $$T : V \to W$$
   that is both one-to-one and onto. In other words, V and W have the same vector space structure.

7. **Dimension Theorem:** If $T : V \to W$ is linear and V is finite-dimensional, then
   $$\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(V).$$
   Equivalently,
   $$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

8. **Definition:** A linear transformation T is **one-to-one** if whenever $T(u) = T(v)$, it follows that $u = v$. Equivalently, T is one-to-one if $\ker(T) = \{0\}$.

9. **Definition:** An **inverse** of a linear transformation $T : V \to W$ is a linear transformation $T^{-1} : W \to V$ such that $T^{-1}T = I_V$ and $TT^{-1} = I_W$. This means applying T and then T^{-1} returns the original vector, and vice versa.

2 Let $\beta = \{e_1, e_2, e_3\}$ be the standard ordered basis of $\mathbb{R}^3$. Define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(a_1, a_2, a_3) = (a_1 - 2a_2 + a_3,\ a_1 + a_2 - 2a_3,\ -2a_1 + a_2 + a_3).$$

Let $\gamma = \{1,\ 1 + x,\ 1 + x + x^2\}$ be an ordered basis of $P_2(\mathbb{R})$. Define another linear transformation

$$U : \mathbb{R}^3 \to P_2(\mathbb{R}), \qquad U(a_1, a_2, a_3) = a_1 + a_2 x + a_3 x^2.$$

(a). Compute $[T]^{\beta}_{\beta}$.

(b). Is T invertible? Justify your answer *(hint: try (1,1,1))*.

(c). Compute $[U]^{\gamma}_{\beta}$.

(d). Compute the matrix of the composition $UT : \mathbb{R}^3 \to P_2(\mathbb{R})$, that is, $[UT]^{\gamma}_{\beta}$.

Solutions: (a). To compute $[T]^{\beta}_{\beta}$, we apply T to the standard basis vectors of $\mathbb{R}^3$.

$$T(e_1) = T(1, 0, 0) = (1, 1, -2),$$
$$T(e_2) = T(0, 1, 0) = (-2, 1, 1),$$
$$T(e_3) = T(0, 0, 1) = (1, -2, 1).$$

Thus the columns of $[T]^{\beta}_{\beta}$ are the coordinate vectors of $T(e_1), T(e_2), T(e_3)$ relative to the standard basis β. Therefore,

$$[T]^{\beta}_{\beta} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}.$$

(b). T is **not invertible**. One way to justify this is to observe that

$$T(1, 1, 1) = (1 - 2 + 1,\ 1 + 1 - 2,\ -2 + 1 + 1) = (0, 0, 0).$$

Since $(1, 1, 1) \neq (0, 0, 0)$ but $T(1, 1, 1) = (0, 0, 0)$, the transformation T has a nonzero vector in its kernel. Hence T is not one-to-one, so T is not invertible.

Equivalently, the rows of $[T]^{\beta}_{\beta}$ add to zero, so its determinant is 0, and therefore the matrix is not invertible.

(c). To compute $[U]^{\gamma}_{\beta}$, we apply U to the standard basis vectors e_1, e_2, e_3 of $\mathbb{R}^3$, and then express the resulting polynomials in the basis $\gamma = \{1,\ 1 + x,\ 1 + x + x^2\}$.

First,

$$U(e_1) = U(1, 0, 0) = 1.$$

Now

$$1 = 1 \cdot 1 + 0 \cdot (1 + x) + 0 \cdot (1 + x + x^2),$$

so

$$[U(e_1)]_\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Next,

$$U(e_2) = U(0,1,0) = x.$$

Write

$$x = c_1 \cdot 1 + c_2(1+x) + c_3(1+x+x^2).$$

Comparing coefficients gives

$$c_1 + c_2 + c_3 = 0, \qquad c_2 + c_3 = 1, \qquad c_3 = 0.$$

So $c_3 = 0$, $c_2 = 1$, and $c_1 = -1$. Hence

$$[U(e_2)]_\gamma = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Finally,

$$U(e_3) = U(0,0,1) = x^2.$$

Write

$$x^2 = c_1 \cdot 1 + c_2(1+x) + c_3(1+x+x^2).$$

Comparing coefficients gives

$$c_1 + c_2 + c_3 = 0, \qquad c_2 + c_3 = 0, \qquad c_3 = 1.$$

So $c_3 = 1$, $c_2 = -1$, and $c_1 = 0$. Therefore

$$[U(e_3)]_\gamma = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Thus,

$$[U]_\beta^\gamma = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d). We use the composition rule:

$$[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta^\beta.$$

From parts (a) and (c),

$$[U]_\beta^\gamma = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [T]_\beta^\beta = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}.$$

So

$$[UT]_{\beta}^{\gamma} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}.$$

Carrying out the multiplication,

$$[UT]_{\beta}^{\gamma} = \begin{bmatrix} 0 & -3 & 3 \\ 3 & 0 & -3 \\ -2 & 1 & 1 \end{bmatrix}.$$

Hence,

$$[UT]_{\beta}^{\gamma} = \begin{bmatrix} 0 & -3 & 3 \\ 3 & 0 & -3 \\ -2 & 1 & 1 \end{bmatrix}.$$

□

3 Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces such that $\dim(V) = \dim(W)$, and let $T : \mathbf{V} \to \mathbf{W}$ be linear. Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

Solutions:

Proof. We will prove both directions.

($\Rightarrow$) Suppose that $S \subseteq V$ is linearly independent. We want to show that $T(S)$ is linearly independent in W.

Let $v_1, \ldots, v_k \in S$ be distinct vectors, and suppose

$$c_1T(v_1) + c_2T(v_2) + \cdots + c_kT(v_k) = 0$$

for some scalars $c_1, \ldots, c_k$.

Since T is linear,

$$T(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = 0.$$

Thus

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k \in \ker(T).$$

Because T is one-to-one, we have

$$\ker(T) = \{0\}.$$

So

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0.$$

But S is linearly independent, so $v_1, \ldots, v_k$ are linearly independent. Therefore,

$$c_1 = c_2 = \cdots = c_k = 0.$$

Hence $T(v_1), \ldots, T(v_k)$ are linearly independent. Since this is true for any finite collection of vectors from S, it follows that $T(S)$ is linearly independent.

($\Leftarrow$) Suppose that $T(S)$ is linearly independent. We want to show that S is linearly independent.

Let $v_1, \ldots, v_k \in S$ and suppose

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

for some scalars $c_1, \ldots, c_k$.

Apply T to both sides. Since T is linear,

$$T(c_1v_1 + \cdots + c_kv_k) = c_1T(v_1) + \cdots + c_kT(v_k) = T(0) = 0.$$

Thus

$$c_1T(v_1) + c_2T(v_2) + \cdots + c_kT(v_k) = 0.$$

Since $T(S)$ is linearly independent, we must have $c_1 = c_2 = \cdots = c_k = 0$. Therefore S is linearly independent. $\square$

4 Let B be an $n \times n$ invertible matrix. Define $\phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\phi(A) = B^{-1}AB$. Prove that ϕ is an isomorphism. **Solution**:

Proof. **Step 1: ϕ is linear.**

Let $A_1, A_2 \in M_{n\times n}(F)$ and $c \in F$. Then

$$\phi(A_1 + cA_2) = B^{-1}(A_1 + cA_2)B = B^{-1}A_1B + cB^{-1}A_2B = \phi(A_1) + c\phi(A_2).$$

Hence ϕ is linear.

Step 2: ϕ is one-to-one.

Suppose $\phi(A) = 0$. Then $B^{-1}AB = 0$. Multiplying on the left by B and on the right by B^{-1}, we get $A = 0$. Thus $\ker(\phi) = \{0\}$, so ϕ is injective.

Step 3: ϕ is onto.

Let $C \in M_{n\times n}(F)$. We want to find $A \in M_{n\times n}(F)$ such that $\phi(A) = C$. Take $A = BCB^{-1}$. Then $\phi(A) = B^{-1}(BCB^{-1})B = C$. Therefore ϕ is surjective.

Since ϕ is linear, injective, and surjective, it is an isomorphism.

In fact, the inverse map is $\phi^{-1}(C) = BCB^{-1}$. □

Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over a field F such that $\dim(\mathbf{V}) = \dim(\mathbf{W}) = n$. Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Suppose there is an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$ of $\mathbf{V}$ such that the set $\gamma = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is an ordered basis of $\mathbf{W}$. Show that $[T]_\beta^\gamma = I_n$, the $n \times n$ identity matrix.

Solution:

Proof. By assumption, $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of $\mathbf{V}$, and $\gamma = \{T(v_1), \ldots, T(v_n)\}$ is an ordered basis of $\mathbf{W}$.

Recall the definition of the matrix of T with respect to β and γ: the jth column of $[T]_\beta^\gamma$ is the coordinate vector of $T(v_j)$ relative to the basis γ.

We now write each $T(v_j)$ as a linear combination of the basis vectors in γ.

But by construction,

$$T(v_j) = T(v_j) = 0 \cdot T(v_1) + \cdots + 0 \cdot T(v_{j-1}) + 1 \cdot T(v_j) + 0 \cdot T(v_{j+1}) + \cdots + 0 \cdot T(v_n).$$

Therefore, the coordinates of $T(v_j)$ with respect to γ are

$$[T(v_j)]_\gamma = e_j,$$

the jth standard basis vector in F^n (a 1 in the jth position and 0 in all others).

Hence, the jth column of $[T]_\beta^\gamma$ is e_j for each $j = 1, \ldots, n$. This means that

$$[T]_\beta^\gamma = I_n.$$

Thus, under the bases β and γ constructed above, the matrix of T is the identity matrix. □

5 Let $\mathbf{V}$ be a finite-dimensional vector space over F, and let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for $\mathbf{V}$. Define the coordinate mapping $\phi_\beta : \mathbf{V} \to F^n$ by $\phi_\beta(v) = [v]_\beta$, where $[v]_\beta$ denotes the coordinate vector of v relative to the basis β.

$$\begin{array}{ccc} \mathbf{V} & \xrightarrow{T} & \mathbf{V} \\ \downarrow{\scriptstyle \phi_\beta} & & \downarrow{\scriptstyle \phi_\beta} \\ F^n & \xrightarrow{[T]_\beta} & F^n \end{array}$$

Prove that ϕ_β is an isomorphism.

Proof. **Step 1: ϕ_β is linear.**

Let $u, v \in \mathbf{V}$ and $c \in F$. Write $u = \sum_{i=1}^n a_i v_i$; $\quad v = \sum_{i=1}^n b_i v_i$. Then $u + cv = \sum_{i=1}^n (a_i + cb_i) v_i$. Hence

$$\phi_\beta(u + cv) = [u + cv]_\beta = \begin{pmatrix} a_1 + cb_1 \\ a_2 + cb_2 \\ \vdots \\ a_n + cb_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + c \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \phi_\beta(u) + c\phi_\beta(v).$$

So ϕ_β is linear.

Step 2: ϕ_β is one-to-one (injective). Suppose $\phi_\beta(v) = 0$. Then $[v]_\beta = 0$, so all coordinates of v relative to β are zero. Thus $v = 0v_1 + \cdots + 0v_n = 0$. Therefore, $\ker(\phi_\beta) = \{0\}$, and so ϕ_β is injective.

Step 3: ϕ_β is onto (surjective).

Let

$$x = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n.$$

Define $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \in \mathbf{V}$. Then by definition of coordinates,

$$\phi_\beta(v) = [v]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = x.$$

So ϕ_β is surjective.

Since ϕ_β is linear, one-to-one, and onto, it is an isomorphism. □

6 **(Bonus question)** Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation from an n-dimensional vector space $\mathbf{V}$ to an m-dimensional vector space $\mathbf{W}$. Let β and γ be ordered bases for $\mathbf{V}$ and $\mathbf{W}$, respectively. Prove that $\text{rank}(T) = \text{rank}(L_A)$ and that $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_\beta^\gamma$.

Proof. it is helpful to use the commutative diagram associated with the matrix representation $A = [T]_\beta^\gamma$:

$$\begin{array}{ccc} \mathbf{V} & \xrightarrow{T} & \mathbf{W} \\ {\scriptstyle \phi_\beta}\downarrow & & \downarrow{\scriptstyle \phi_\gamma} \\ F^n & \xrightarrow{L_A} & F^m \end{array}$$

Since $\phi_\beta : \mathbf{V} \to F^n$ and $\phi_\gamma : \mathbf{W} \to F^m$ are isomorphisms, they preserve subspaces and dimensions. We first prove that $\text{rank}(T) = \text{rank}(L_A)$.

Recall that

$$R(T) = \{T(v) : v \in \mathbf{V}\} \quad \text{and} \quad R(L_A) = \{L_A(x) : x \in F^n\}.$$

From the commutative relation $\phi_\gamma \circ T = L_A \circ \phi_\beta$, if $y \in R(T)$, then $y = T(v)$ for some $v \in \mathbf{V}$, and hence $\phi_\gamma(y) = \phi_\gamma(T(v)) = L_A(\phi_\beta(v)) \in R(L_A)$. Thus $\phi_\gamma(R(T)) \subseteq R(L_A)$.

Conversely, if $z \in R(L_A)$, then $z = L_A(x)$ for some $x \in F^n$. Since ϕ_β is onto, there exists $v \in \mathbf{V}$ such that $x = \phi_\beta(v)$. Therefore $z = L_A(\phi_\beta(v)) = \phi_\gamma(T(v)) \in \phi_\gamma(R(T))$. So $R(L_A) \subseteq \phi_\gamma(R(T))$. Hence

$$\phi_\gamma(R(T)) = R(L_A).$$

Because ϕ_γ is an isomorphism, Exercise 17 implies that $\dim(R(T)) = \dim(\phi_\gamma(R(T)))$. Therefore

$$\text{rank}(T) = \dim(R(T)) = \dim(R(L_A)) = \text{rank}(L_A).$$

Now we prove that $\text{nullity}(T) = \text{nullity}(L_A)$.

Recall that $N(T) = \{v \in \mathbf{V} : T(v) = 0\}$ and $N(L_A) = \{x \in F^n : L_A(x) = 0\}$. Let $v \in N(T)$. Then $T(v) = 0$, so by the commutative relation, $L_A(\phi_\beta(v)) = \phi_\gamma(T(v)) = \phi_\gamma(0) = 0$. Thus $\phi_\beta(v) \in N(L_A)$, and hence $\phi_\beta(N(T)) \subseteq N(L_A)$.

Conversely, let $x \in N(L_A)$. Since ϕ_β is onto, there exists $v \in \mathbf{V}$ such that $x = \phi_\beta(v)$. Then

$$0 = L_A(x) = L_A(\phi_\beta(v)) = \phi_\gamma(T(v)).$$

Because ϕ_γ is one-to-one, this implies $T(v) = 0$. Hence $v \in N(T)$, so $x = \phi_\beta(v) \in \phi_\beta(N(T))$. Therefore

$$N(L_A) = \phi_\beta(N(T)).$$

Again, since ϕ_β is an isomorphism, Exercise 17 gives $\dim(N(T)) = \dim(\phi_\beta(N(T))) = \dim(N(L_A))$. Thus

$$\text{nullity}(T) = \text{nullity}(L_A).$$

Therefore,

$$\boxed{\text{rank}(T) = \text{rank}(L_A)} \quad \text{and} \quad \boxed{\text{nullity}(T) = \text{nullity}(L_A)}.$$

□

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