UF MAS 4105 **Exam 2**

Name: _____

- Explain your work (efficiently); partial credit is available.
- No notes, books, calculators, or other electronic devices are permitted.
- Please sign below to indicate you accept the following statement:

  "I will not give, receive, or use any unauthorized assistance."

Signature: _____

Problem	**Total Points**	**Score**
1	10	
2	10	
3	10	
4	10	
5	10	
Total	**50**	

UF MAS 4105 **Exam 1 B**

1 Judge whether each of the following statements for (1)-(8) is **true** or **false**.

(1). The empty set is linearly dependent.

(2). If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \cdots, x_n$ are linear independent, then all the scalars a_i are zero.

(3). Every system of linear equations has a solution.

(4). The dimension of $\mathbf{M}_{m\times n}(F)$ is $m + n$.

(5). If S generates the vector space $\mathbf{V}$, then every vector in $\mathbf{V}$ can be written as a linear combination of vectors in S in only one way.

(6). The zero vector space has no basis.

(7). The span of $\emptyset$ is $\emptyset$

(8). Infinite-dimensional vector space can have a finite basis.

Solutions:

1. **False.** A set of vectors is *linearly dependent* if there exists a *nontrivial* linear combination of finitely many vectors in the set that equals the zero vector. For the empty set, there are no vectors and hence no nontrivial linear combinations at all. By convention, the empty set is linearly *independent*.

2. **True.** This is exactly the definition of linear independence: a set $\{x_1, \ldots, x_n\}$ is linearly independent if
$$a_1x_1 + \cdots + a_nx_n = 0$$
implies $a_1 = \cdots = a_n = 0$.

3. **False.** A system of linear equations can be inconsistent. For example,
$$\begin{cases} x = 0, \\ x = 1 \end{cases}$$
has no solution.

4. **False.** An $m \times n$ matrix over F has mn entries, and a standard basis is given by the mn matrices E_{ij} with a 1 in the (i, j)-position and 0 elsewhere. Thus
$$\dim \mathbf{M}_{m\times n}(F) = mn,$$
not $m + n$.

5. **False.** If S *generates* (spans) V, then every vector in V can be written as a linear combination of vectors in S, but not necessarily in a unique way. Uniqueness of representation holds when S is a *basis* (i.e., spanning and linearly independent), not for an arbitrary generating set.

6. **False.** The zero vector space $\{0\}$ has dimension 0. By convention, its basis is the empty set $\emptyset$. So it *does* have a basis (the unique one), and that basis has 0 elements.

7. **False.** The span of a set is the set of all finite linear combinations of its vectors. For the empty set, the only linear combination is the empty sum, which is defined to be the zero vector. Hence
$$\text{span}(\emptyset) = \{0\},$$
not $\emptyset$.

8. **False.** If a vector space has a *finite* basis with n elements, then its dimension is n and it is called *finite-dimensional*. An infinite-dimensional vector space is defined precisely as one that does not admit any finite basis. Therefore, an infinite-dimensional vector space cannot have a finite basis.

2 Let u, v be distinct vectors in a vector space V such that $\{u, v\}$ is a basis for V. Let $a, b, c, d \in \mathbb{F}$ be scalars with $a \neq 0$ and $ad - bc \neq 0$. Show that the set $\{\, au + bv,\ cu + dv \,\}$ is a basis for V.

Solution:

Proof. Since $\{u, v\}$ is a basis for V, the vectors u and v are linearly independent and $\dim V = 2$.

(Show that $\{au + bv,\ cu + dv\}$ is a basis when $ad - bc \neq 0$.

Because V is 2-dimensional, it suffices to show that $au + bv$ and $cu + dv$ are linearly independent.

Suppose

$$\alpha(au + bv) + \beta(cu + dv) = 0$$

for some scalars α, β. Expanding, we obtain

$$(\alpha a + \beta c)u + (\alpha b + \beta d)v = 0.$$

Since $\{u, v\}$ is linearly independent, the coefficients of u and v must both be zero:

$$\begin{cases} \alpha a + \beta c = 0, \\ \alpha b + \beta d = 0. \end{cases}$$

This is a homogeneous linear system in α and β with coefficient matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

The determinant of this matrix is

$$\det = ad - bc.$$

By assumption, $ad - bc \neq 0$, so the system has only the trivial solution

$$\alpha = 0, \quad \beta = 0.$$

Therefore $\{au + bv,\ cu + dv\}$ is linearly independent. Since it contains two linearly independent vectors in a 2-dimensional space, it is a basis for V.

□

3 Do the polynomials $x^3 - 2x^2 + 1, 4x^2 - x + 3$, and $3x - 2$ generate $\mathbf{P}_3(\mathbb{R})$? Justify your answer.

Solution:

Proof. Let the polynomials $p_1(x) = x^3 - 2x^2 + 1, \quad p_2(x) = 4x^2 - x + 3, \quad p_3(x) = 3x - 2$. The vector space $\mathbf{P}_3(\mathbb{R})$ consists of all real polynomials of degree at most 3 and has dimension

$$\dim \mathbf{P}_3(\mathbb{R}) = 4,$$

with standard basis $\{1, x, x^2, x^3\}$.

Any spanning set (generating set) of a finite-dimensional vector space must have at least as many elements as the dimension of the space. Since we are given only 3 polynomials, the subspace they span has dimension at most 3, and therefore cannot equal all of $\mathbf{P}_3(\mathbb{R})$, which has dimension 4.

Hence $\{p_1, p_2, p_3\}$ does not generate $\mathbf{P}_3(\mathbb{R})$. □

4 Let S be a set of nonzero polynomials $\mathbf{P}(F)$ such that no two have the same degree. Prove that S is linearly independent.

Solution:

Proof. We prove this by contradiction. Suppose S is not linearly independent. Then there exists a finite subset $\{p_1, p_2, \ldots, p_m\} \subseteq S$ that is linearly dependent. Since no two polynomials in S have the same degree and all are nonzero, we may reorder the polynomial index to get $\{p_1, p_2, \ldots, p_m\} \subseteq S$ with $\deg p_1 < \deg p_2 < \cdots < \deg p_m$.

Since $\{p_1, \ldots, p_m\}$ is linearly dependent, there exist not all zero scalars $a_1, \ldots, a_m \in F$, s.t.

$$a_1 p_1 + a_2 p_2 + \cdots + a_m p_m = 0,$$

where the right-hand side is the zero polynomial.

Consider the polynomial on the left-hand side. Let $\deg p_m = d$, and let the leading coefficient of p_m be $c \neq 0$. Then the term of highest degree in the sum

$$a_1 p_1 + \cdots + a_m p_m$$

is the x^d-term coming from $a_m p_m$, since all other p_i have strictly smaller degree. The coefficient of x^d in the sum is therefore $a_m c$. But the sum equals the zero polynomial, whose coefficient of x^d is 0. Hence

$$a_m c = 0 \quad \Rightarrow \quad a_m = 0,$$

since $c \neq 0$.

Now the relation reduces to

$$a_1 p_1 + \cdots + a_{m-1} p_{m-1} = 0.$$

Applying the same argument to this shorter relation (again looking at the largest degree among $p_1, \ldots, p_{m-1}$), we conclude $a_{m-1} = 0$. Repeating this step, we eventually obtain

$$a_1 = a_2 = \cdots = a_m = 0.$$

This contradicts our assumption that the relation was nontrivial. Hence every finite subset of S is linearly independent, and thus S is linearly independent. □

5 For a fixed $a \in \mathbb{R}$, determine the dimension of the subspace of $\mathbf{P}_n(\mathbb{R})$ defined by $\{f \in \mathbf{P}_n(\mathbb{R}) : f(a) = 0\}$.

Solution: We consider the subspace

$$U = \{f \in \mathbf{P}_n(\mathbb{R}) : f(a) = 0\},$$

where $a \in \mathbb{R}$ is fixed.

Proof. Recall that $\mathbf{P}_n(\mathbb{R})$ consists of all real polynomials of degree at most n and has dimension $n+1$, with standard basis

$$\{1, x, x^2, \ldots, x^n\}.$$

First, note that if $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$. Thus every polynomial $f \in U$ can be written in the form

$$f(x) = (x-a)g(x),$$

where $g(x)$ is a polynomial of degree at most $n-1$.

Conversely, for any polynomial $g \in \mathbf{P}_{n-1}(\mathbb{R})$ of degree at most $n-1$, the polynomial $f(x) = (x-a)g(x)$ satisfies $f(a) = 0$, so $f \in U$. Hence every element of U arises in this way.

Now consider the set

$$\mathcal{B} = \{(x-a), (x-a)x, (x-a)x^2, \ldots, (x-a)x^{n-1}\}.$$

Each polynomial in $\mathcal{B}$ belongs to U, and there are exactly n of them.

We claim that $\mathcal{B}$ is a basis for U.

Spanning. Let $f \in U$. Then $f(x) = (x-a)g(x)$ for some $g(x) \in \mathbf{P}_{n-1}(\mathbb{R})$. Since $\{1, x, x^2, \ldots, x^{n-1}\}$ is a basis for $\mathbf{P}_{n-1}(\mathbb{R})$, we may write

$$g(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}.$$

Thus

$$f(x) = (x-a)(c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}),$$

which is a linear combination of the elements of $\mathcal{B}$. Hence $\mathcal{B}$ spans U.

Linear independence. Suppose

$$c_0(x-a) + c_1(x-a)x + \cdots + c_{n-1}(x-a)x^{n-1} = 0.$$

Factoring out $(x-a)$ gives

$$(x-a)\big(c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}\big) = 0.$$

Since $(x-a)$ is not the zero polynomial, the factor in parentheses must be the zero polynomial. Thus

$$c_0 = c_1 = \cdots = c_{n-1} = 0.$$

Hence the set $\mathcal{B}$ is linearly independent.

Therefore $\mathcal{B}$ is a basis for U, and since it has n elements, we conclude that

$$\dim U = n.$$

□

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