UF MAS 4105 **Exam 1 A**

Name: _____

- Explain your work (efficiently); partial credit is available.
- No notes, books, calculators, or other electronic devices are permitted.
- Please sign below to indicate you accept the following statement:

  "I will not give, receive, or use any unauthorized assistance."

Signature: _____

Problem	**Total Points**	**Score**
1	10	
2	10	
3	10	
4	10	
5	10	
Total	**50**	

UF MAS 4105 **Exam 1 A**

1 Judge whether each of the following statements for (1)-(4) is **true** or **false**. Give a brief reason.

(1). A vector space may have more than one zero vector.

(2). In any vector space, $a\mathbf{x} = a\mathbf{y}$ implies that $\mathbf{x} = \mathbf{y}$.

(3). Every system of linear equations has a solution.

(4). The intersection of any two subsets of a vector space $\mathbf{V}$ is a subspace of $\mathbf{V}$.

For the following, determine the transpose of each of the matrices below. If the matrix is square, also compute its trace.

$$\text{(i)} \quad \begin{pmatrix} -4 & 2 & 6 \\ 5 & -1 & -3 \\ 6 & -3 & 5 \end{pmatrix}, \qquad \text{(ii)} \quad \begin{pmatrix} 2026 & -2025 & 2024 & 2023 \end{pmatrix}.$$

Solutions:

(1) **False.** In a vector space, the zero vector is unique. If $\mathbf{0}$ and $\mathbf{0}'$ both satisfy $x + \mathbf{0} = x$ and $x + \mathbf{0}' = x$ for all x, then $\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'$, so there cannot be more than one zero vector.

(2) **False in general.** If $a \neq 0$, then $a\mathbf{x} = a\mathbf{y}$ implies $\mathbf{x} = \mathbf{y}$. But if $a = 0$, then $a\mathbf{x} = a\mathbf{y}$ becomes $\mathbf{0} = \mathbf{0}$ for every $\mathbf{x}, \mathbf{y}$. Example: take $a = 0$ and $\mathbf{x} \neq \mathbf{y}$; then $a\mathbf{x} = a\mathbf{y}$, but $\mathbf{x} \neq \mathbf{y}$.

(3) **False.** Some systems are inconsistent, for example

$$x = 0, \qquad x = 1.$$

No x satisfies both equations, so this system has no solution.

(4) **False.** The intersection of two arbitrary subsets of V need not be a subspace. For instance, in $\mathbb{R}^2$ let

$$A = \{(1, 0)\}, \qquad B = \{(1, 1)\}.$$

Then $A \cap B = \varnothing$, which is not a subspace since it does not contain the zero vector.

Transpose and trace.

(i) Let

$$A = \begin{pmatrix} -4 & 2 & 6 \\ 5 & -1 & -3 \\ 6 & -3 & 5 \end{pmatrix}. \quad A^T = \begin{pmatrix} -4 & 5 & 6 \\ 2 & -1 & -3 \\ 6 & -3 & 5 \end{pmatrix},$$

$$\operatorname{tr}(A) = -4 + (-1) + 5 = 0.$$

(ii)

$$B^T = \begin{pmatrix} 2026 \\ -2025 \\ 2024 \\ 2023 \end{pmatrix}.$$

□

2 Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in \mathbf{V}$ and $c \in \mathbb{R}$,define:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2), \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Using the vector space axioms VS(3), and VS(8), show that $\mathbf{V}$ is *not* a vector space over $\mathbb{R}$ under these operations.

Remark. the statement of VS(3), VS(8) is as below:

VS(3): There exists an element in $\mathbf{V}$ denoted by 0 such that $x+0 = x$ for each x in $\mathbf{V}$.

VS(8): For each pair of element a, b in $\mathbb{R}$ and each element x in $\mathbf{V}$, $(a + b)x = ax + bx$.

Solution:

Suppose (e_1, e_2) is an additive identity in $\mathbf{V}$. Then for all $(a_1, a_2) \in \mathbf{V}$ we must have

$$(a_1, a_2) + (e_1, e_2) = (a_1, a_2) \quad \text{and} \quad (e_1, e_2) + (a_1, a_2) = (a_1, a_2).$$

From

$$(a_1, a_2) + (e_1, e_2) = (a_1 + 2e_1,\ a_2 + 3e_2) = (a_1, a_2),$$

we obtain

$$a_1 + 2e_1 = a_1 \Rightarrow 2e_1 = 0 \Rightarrow e_1 = 0,$$
$$a_2 + 3e_2 = a_2 \Rightarrow 3e_2 = 0 \Rightarrow e_2 = 0.$$

So any additive identity must be $(0, 0)$.

Now check whether $(0, 0)$ actually behaves as an identity on the left:

$$(0, 0) + (a_1, a_2) = (0 + 2a_1,\ 0 + 3a_2) = (2a_1, 3a_2).$$

For $(0, 0)$ to be an additive identity, we would need

$$(2a_1, 3a_2) = (a_1, a_2) \quad \text{for all } (a_1, a_2) \in \mathbf{V},$$

which implies

$$2a_1 = a_1, \quad 3a_2 = a_2 \quad \text{for all } a_1, a_2 \in \mathbb{R}.$$

This would force $a_1 = 0$ and $a_2 = 0$ for all vectors, which is false.

Thus there is no vector in $\mathbf{V}$ that serves as an additive identity for all elements of $\mathbf{V}$. Since the additive identity axiom fails, $\mathbf{V}$ is not a vector space over $\mathbb{R}$ with these operations.

In a vector space, scalar multiplication must satisfy

$$c(u + v) = cu + cv \quad \text{for all } c \in \mathbb{R},\ u, v \in \mathbf{V}.$$

Let $u = (1, 1)$, $v = (1, 1)$, and $c = 1$. Then

$$u + v = (1 + 2 \cdot 1,\ 1 + 3 \cdot 1) = (3, 4).$$

Hence,

$$c(u+v) = 1(3,4) = (3,4).$$

On the other hand,

$$cu = (1,1), \qquad cv = (1,1),$$

and thus

$$cu + cv = (1,1) + (1,1) = (1 + 2 \cdot 1,\ 1 + 3 \cdot 1) = (3,4).$$

So distributivity holds in this case. However, scalar multiplication must also satisfy

$$(c+d)v = cv + dv \quad \text{for all } c, d \in \mathbb{R}.$$

Let $v = (1,1)$ and choose $c = d = 1$. Then

$$(c+d)v = 2(1,1) = (2,2),$$

while

$$cv + dv = (1,1) + (1,1) = (3,4).$$

Since

$$(2,2) \neq (3,4),$$

the axiom $(c+d)v = cv + dv$ fails.

Therefore, $\mathbf{V}$ is not a vector space over $\mathbb{R}$. □

3 Let $P(F)$ denote the vector space of all polynomials over the field F. Is the set $W = \{f(x) \in P(F) : f(x)$ has degree less and equal to $n\}$ a subspace of $P(F)$ if $n \geq 0$? If yes, give the proof. If no, justify your answer.

Solutions:

Let $P(F)$ be the vector space of all polynomials over F, and

$$W = \{f(x) \in P(F) : \deg f(x) \leq n\}, \quad n \geq 0.$$

We check the subspace conditions.

(1) Nonempty. The zero polynomial 0 is in $P(F)$, and by convention $\deg 0 \leq n$. Hence $0 \in W$.

(2) Closed under addition. Let $f, g \in W$. Then $\deg f \leq n$ and $\deg g \leq n$. We have

$$\deg(f + g) \leq \max\{\deg f, \deg g\} \leq n,$$

so $f + g \in W$.

(3) Closed under scalar multiplication. Let $f \in W$ and $c \in F$. If $c = 0$, then $cf = 0 \in W$. If $c \neq 0$, then $\deg(cf) = \deg(f) \leq n$, so $cf \in W$.

Thus W is nonempty and closed under addition and scalar multiplication. Therefore W is a subspace of $P(F)$. □

4 Solve the following systems of linear equations

$$\begin{cases} x_1 + 2x_2 + 2x_3 = 2 \\ x_1 + 8x_3 + 5x_4 = -6 \\ x_1 + x_2 + 5x_3 + 5x_4 = 3. \end{cases} \tag{1}$$

Solutions:

We subtract the first equation from the second and from the third to eliminate x_1.

From $\text{Eq}_2 - \text{Eq}_1$:

$$\begin{aligned} (x_1 + 8x_3 + 5x_4) - (x_1 + 2x_2 + 2x_3) &= -6 - 2, \\ -2x_2 + 6x_3 + 5x_4 &= -8. \end{aligned} \tag{A}$$

From $\text{Eq}_3 - \text{Eq}_1$:

$$\begin{aligned} (x_1 + x_2 + 5x_3 + 5x_4) - (x_1 + 2x_2 + 2x_3) &= 3 - 2, \\ -x_2 + 3x_3 + 5x_4 &= 1. \end{aligned} \tag{B}$$

Now solve (A) and (B) for x_2 and x_4 in terms of x_3. From (B),

$$-x_2 + 3x_3 + 5x_4 = 1 \quad \Rightarrow \quad x_2 = 3x_3 + 5x_4 - 1.$$

Substitute this expression for x_2 into (A):

$$\begin{aligned} -2(3x_3 + 5x_4 - 1) + 6x_3 + 5x_4 &= -8, \\ -6x_3 - 10x_4 + 2 + 6x_3 + 5x_4 &= -8, \\ -5x_4 + 2 &= -8, \end{aligned}$$

so

$$-5x_4 = -10 \quad \Rightarrow \quad x_4 = 2.$$

Then from (B),

$$-x_2 + 3x_3 + 5 \cdot 2 = 1 \quad \Rightarrow \quad -x_2 + 3x_3 + 10 = 1 \quad \Rightarrow \quad x_2 = 3x_3 + 9.$$

Finally, substitute x_2 and x_4 into the first equation:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + 2(3x_3 + 9) + 2x_3 &= 2, \\ x_1 + 6x_3 + 18 + 2x_3 &= 2, \\ x_1 + 8x_3 + 18 &= 2, \end{aligned}$$

hence

$$x_1 = -8x_3 - 16.$$

Let t be a free parameter, with $t \in \mathbb{R}$ and $t = x_3$. Then the solution set is

$$(x_1, x_2, x_3, x_4) = (-8t - 16,\ 3t + 9,\ t,\ 2), \qquad t \in \mathbb{R}.$$

5 Find the equations of the planes containing the following points in space, then determine whether the first vector can be expressed as a linear combination of the other two.

$$(2,-1,0),(1,2,-3),(1,-3,2)$$

Solutions

(1) Parametric equation of the plane. Define the direction vectors

$$\mathbf{u} = P_2 - P_1 = (-1,3,-3), \qquad \mathbf{v} = P_3 - P_1 = (-1,-2,2).$$

Hence, the equation for plane containing the three points is given by

$$\mathbf{x} = (2,-1,0) + s(-1,3,-3) + t(-1,-2,2), \qquad s,t \in \mathbb{R}.$$

(2) Linear combination. We now determine whether

$$(2,-1,0) = a(1,2,-3) + b(1,-3,2)$$

for some $a, b \in \mathbb{R}$. This gives the system

$$\begin{aligned} a + b &= 2, \\ 2a - 3b &= -1, \\ -3a + 2b &= 0. \end{aligned}$$

Solving the first two equations yields $a = 1$ and $b = 1$. Substituting into the third equation gives

$$-3(1) + 2(1) = -1 \neq 0,$$

which is a contradiction.

Therefore, $(2,-1,0)$ *cannot* be expressed as a linear combination of $(1,2,-3)$ and $(1,-3,2)$.